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Gentle Perturbations of the Free Bose Gas. I

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It is demonstrated that the thermal structure of the noncritical free Bose gas is completely described by certain periodic generalized Gaussian stochastic process or equivalently by a certain periodic generalized Gaussian random field. Elementary properties of this Gaussian stochastic thermal structure are established. Gentle perturbations of several types of the free thermal stochastic structure are studied. In particular, new models of non-Gaussian thermal structures are constructed and a new functional integral representation of the corresponding Euclidean-time Green functions is obtained rigorously.

KEY WORDS: Free Bose gas; W^*-KMS structure; periodic generalized stochastic process; gentle perturbations; multitime Green functions.

1. INTRODUCTION

A variety of existence and analycity results—as well as constructive ones—have been rigorously obtained for some realistic models of nonrelativistic quantum matter in thermal equilibrium.[1-8] Nevertheless, a number of basic questions on the origin of fundamental macroscopic quantum phenomena such as superconductivity, superfluidity, etc.[9,10] are lacking rigorous demonstration in the above realistic treatments. Only for mean-field-like and exactly solvable models has a mathematically well-defined analysis of these phenomena been performed.[11-13] It is worthwhile to mention here the recent activity on the superconductivity problem in Fermi matter models of physical interest,[14,15] which is based on the rigorous renormalization group approach of Gallavotti and co-workers.[16]

The main objective of the present series of papers is to construct a class of models of self-interacting nonrelativistic Bose matter in a thermal

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equilibrium for which a rigorous discussion of the Bose–Einstein condensation, as well as other phase transitions, would be feasible. In order to approach this goal, we intend to use extensively methods from the constructive Euclidean QFT. In the first paper of the planned series the stochastic content of the fundamental W^*-KMS structure of a free, noncritical Bose gas[17] is described. We prove that the Abelian sector of the Weyl algebra may be described by a certain generalized periodic stochastic process with values in $\mathscr{D}'(\mathbb{R}^d)$ (the space of the Schwartz distributions) and, what is more, that a reconstruction of the whole thermal structure can be derived from it (Proposition 2.5 below). A similar situation occurs in the case of the critical Bose gas when the underlying process is nonergodic.[18] Having described a free Bose gas in terms of stochastic processes, one may perturb them with multiplicative (-like) functionals, thereby creating some new non-Gaussian thermal processes. Furthermore, given such a process, one is able to reproduce its W^*-KMS counterpart by methods of refs. 17, 19, and 20. In this article we shall confine ourselves to the simplest case of perturbations, which we have called (after ref. 21) gentle perturbations of a free thermal process. Using standard tools of statistical mechanics,[22] such as, for example, the Kirkwood–Salsburg analysis, the correlation inequalities of ref. 5, and homogeneous limits, we provide a class of Euclidean invariant models of self-interacting Bose matter than can be controlled rigorously, as we shall demonstrate in Section 3.

The unbounded (of polynomial type) perturbations of a free thermal structure will be studied in another paper of this series.[18] In the critical region nonergodicity is preserved under gentle perturbations (cf. the second part of ref. 18), but whether this is related to the arising of the Bose condensate in an interacting system remains to be resolved.

The pioneering paper of ref. 21 and refs. 23–25 have provided, among others, the major inspirations for our own Euclidean attitude to many-boson physics. The methods of classical statistical mechanics have been already applied to the study of certain quantum systems in refs. 24, 26, and 27, and, to some extent, our approach to an interacting Bose gas resembles that of these authors.

2. FREE BOSE GAS(ES). EUCLIDEAN ASPECTS

The main aim of this section is to point out certain stochastic aspects that arise in the Euclidean time of the thermal structure describing systems of noninteracting Bose particles in the thermal equilibrium at (inverse) temperature $\beta > 0$ and with chemical activity z. Most of the results obtained below apply well to the case when the kinetic energy function $\mathscr{E}(p)$ of the individual particle is such that:

(i) $\forall_{t\in\mathbb{R}_+} e^{-t\mathscr{E}(p)}$ is a positive-definite, continuous function of $p \in \mathbb{R}^d$, or equivalently:

(i') $\{e^{-t\mathscr{E}(-i\nabla)}, t \geqslant 0\}$ generates a semigroup of positivity-preserving operators on $L_2(\mathbb{R}^d)$.

The most general form of such functions is given by the Levi–Khintchine formula (see, e.g., refs. 28 and 29)

$$\mathscr{E}(p) = a + i\mathbf{b}\cdot p + p\cdot C\cdot p - \int [e^{ipx} - 1 - iph(x)]\, r(dx) \tag{2.1}$$

where a is some real constant, $\mathbf{b}$ is some vector in $\mathbb{R}^d$; C is some nonnegative-definite matrix, and r is some nonnegative measure on $\mathbb{R}^d$, called the Levy measure, such that $\int_{\mathbb{R}^d} 1 \wedge |x|^2\, r(dx) < \infty$, where $x \wedge y \equiv \min\{x, y\}$; h is the so-called cutoff function with compact support and satisfying $h(x) = x$ in some neighborhood of the origin (see, e.g., ref. 28 for the role played by the cutoff function h in this scheme). In particular, the functions $\mathscr{E}(p) = |p|^\alpha$, $0 < \alpha \leqslant 2$, or $\mathscr{E}(p) = (p^2 + m^2)^{1/2}$ belong to this class. The common feature of all such functions is that the corresponding semigroups $\{e^{-t\mathscr{E}(-i\nabla)}, t \geqslant 0\}$ are generated by stochastic Markov processes with stationary independent increments known as Levy processes.[28, 29]

The kernels of the semigroups $\{e^{-t\mathscr{E}(-i\nabla)}, t \geqslant 0\}$, denoted as $\mathscr{K}_t^{(\mathscr{E})}(x, y)$, have explicit expressions through the corresponding path space integrals.[29] This enables us to apply the methods of ref. 1 to reproduce (up to some extent) the basic results of refs. 1–4 for interacting gases with nonstandard kinetic energy. The corresponding results are reported elsewhere.[30]

In the present paper we confine ourselves to the following choices:

- $\mathscr{E}(p) = p^2$, called the standard Bose gas.
- $\mathscr{E}(p) = (p^2 + m^2)^{1/2}$, $m \geqslant 0$, called the semirelativistic Bose gas.

In the case of standard Bose gas the corresponding path space integral is well known as the Wiener (conditioned) integral and in this case the corresponding transition function has a kernel

$$\mathscr{K}_t^s(x, y) = \frac{1}{(4\pi t)^{d/2}}\, e^{-|x-y|^2/(4t)^{1/2}} \tag{2.2}$$

In the case of the semirelativistic Bose gas the corresponding transition function has a kernel

$$\mathscr{K}_t^m(x, y) = \frac{4}{\pi^{1/2}} \int_0^\infty d\tau\, \mathscr{K}_\tau^s(x - y)\, \frac{e^{-t^2/4\tau} e^{-m^2\tau}}{\tau^{3/2}} \tag{2.3}$$

with fast exponential decay as $|x-y| \nearrow \infty$ for $m>0$ and in the case $m=0$ equal to the well-known symmetric Cauchy density:

$$\mathscr{K}_t^0(x, y) = \frac{c \cdot t}{(t^2 + |x-y|^2)^{(d+1)/2}} \tag{2.4}$$

2.1. Global Aspects

Let $\mathscr{W}(\mathbf{h})$ be the abstract Weyl algebra built over the one-particle space $\mathbf{h} \equiv L_2(\mathbb{R}^d)$ equipped with the standard symplectic form $\sigma(f, g) \equiv \operatorname{Im}\langle f \mid g\rangle$. For a chosen kinetic energy function $\mathscr{E}(p)$ as above, we define the free thermal state $\omega_0^{(\beta,\mu)}$ on the algebra $\mathscr{W}(\mathbf{h})$:

$$\omega_0^{(\beta,\mu)}(W(f)) \equiv \exp -\tfrac{1}{2} \int dp\, |\hat{f}(p)|^2\, \hat{C}_0^\beta(p) \tag{2.5}$$

where

$$\hat{C}_0^\beta(p) \equiv \frac{1 + ze^{-\beta\mathscr{E}(p)}}{1 - ze^{-\beta\mathscr{E}(p)}} \tag{2.6}$$

$0<\beta$ is the (inverse) temperature, $z \equiv e^{-\beta\mu}$ is the chemical activity, and μ is the chemical potential. The values of z (corresponding to the noncritical regime of the free Bose gas exclusively considered here) are restricted to

$$0 = \sup_p ze^{-\beta\mathscr{E}(p)} < 1$$

which in the case $\mathscr{E}(p) = p^2$ or $\mathscr{E}(p) = |p|$ corresponds to $0<z<1$ (resp. $\mu>0$) and $0<ze^{-\beta m}<1$ (resp. $\mu > -m$) if $m>0$ and $\mathscr{E}(p) = (p^2+m^2)^{1/2}$.

Some elementary properties of the free thermal kernel $C_0^\beta(x)$ are collected in the following proposition.

Proposition 2.1. For any noncritical value of z the corresponding free thermal kernels $C_0^\beta(x)$ have the following properties:

(i) $C_0^\beta(x) = \delta(x) + R_0^\beta(x)$, where $R_0^\beta(x) > 0$ for any $x \in \mathbb{R}^d$ and $R_0^\beta(x) \in S(\mathbb{R}^d)$ if $\mathscr{E}(p) = p^2$ or $\mathscr{E}(p) = (p^2+m^2)^{1/2}$ with $m>0$.

(ii) If $\mathscr{E}(p) = |p|$, then $C_0^\beta(x) = \delta(x) + R_0^\beta(x)$, where $R_0^\beta(x) > 0$ and $R_0^\beta \in C_0(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$.

Proof. From the assumption $\sup_p ze^{-\beta\mathscr{E}(p)} < 1$ we obtain the equality

$$\hat{C}_0^\beta(p) = 1 + \hat{R}_0^\beta(p) \tag{2.7}$$

where $\hat{R}_0^\beta(p) = 2\sum_{n=1}^\infty z^n e^{-\beta n\mathscr{E}(p)}$.

From the positive-definiteness of the function $p \in \mathbb{R}^d \to e^{-it\mathscr{E}(p)}$ for each $t > 0$, it follows that for each n, $\exp[-\beta n\mathscr{E}(p)]$ is the Fourier transform of some positive measure $d\mu_n^\beta$ on $\mathbb{R}^d$. Moreover, from the fact that $\exp - \beta n\mathscr{E}(p) \in S(\mathbb{R}^d)$ in case (i) it follows that $d\mu_n^\beta(x) = \rho_n^\beta(x)\, d^dx$, with $\rho_n^\beta(x) \in S(\mathbb{R}^d)$. By elementary arguments it follows that also $\sum_{n=1}^\infty z^n \exp - \beta n\mathscr{E}(p) \in S(\mathbb{R}^d)$ in case (i); therefore we conclude that all assertions of (i) are valid. The conclusions of (ii) follow from the explicit form (2.4) of the corresponding kernels and elementary arguments. ∎

Let $(\mathscr{H}_0, \Omega_0, \pi_0)$ be the corresponding GNS triplet obtained from $(\mathscr{W}(\mathbf{h}), \omega_0^{(\beta,\mu)})$. Then defining $\alpha_t^0(\pi_0(W(f)) \equiv \pi_0(W(z^{-it/\beta} e^{it\mathscr{E}(p)} f))$, we obtain a σ-weakly continuous group of automorphisms of $\pi_0(\mathscr{W}(\mathbf{h}))$." It is well known that the system $\mathbb{C}_0 \equiv (\mathscr{H}_0, \Omega_0, \alpha_t^0; \pi_0(\mathscr{W}(\mathbf{h}))'')$ forms a W^*-KMS system in the (inverse) temperature β (see, e.g., ref. 17). The corresponding multitime Green functions of the system $\mathbb{C}_0$ are given by

$$\begin{aligned} G_0((t_1, f_1),..., (t_n, f_n)) &\equiv \omega_0^{(\beta,\mu)}(\alpha_{t_1}^0(\pi_0(W(f_1)) \cdots \alpha_{t_n}^0(\pi_0(W(f_n))) \\ &= \prod_{1 \leqslant i \leqslant j \leqslant n} \Big[\exp i\sigma((t_i, f_i), (t_j, f_j)) \\ &\quad \times \exp - \tfrac{1}{2} \int \overline{\hat{f}_i(p)}\, \hat{f}_j(p)\, \hat{G}_0^\beta(t_i - t_j; p)\, d\mathbf{p} \Big] \end{aligned} \tag{2.8}$$

where

$$\sigma((t_i, f_i), (t_j, f_j)) = \operatorname{Im}\langle z^{-it_i/\beta} e^{it_i\mathscr{E}(p)} \hat{f}_i \mid z^{-it_j/\beta} e^{it_j\mathscr{E}(p)} \hat{f}_j \rangle \tag{2.9}$$

$$\hat{G}_0^\beta(t; p) \equiv \frac{z^{-it/\beta} e^{it\mathscr{E}(p)} + z^{1+it/\beta} e^{-(\beta+it)\mathscr{E}(p)}}{1 - ze^{-\beta\mathscr{E}(p)}} \tag{2.10}$$

By elementary arguments they can be extended analytically to the holomorphic functions $G_0((\zeta_1, f_1),..., (\zeta_n, f_n))$ of

$$\begin{aligned} \zeta = (\zeta_1,..., \zeta_n) \in T_n^\beta \\ \equiv \Big\{ \zeta^n = (\zeta_1,..., \zeta_n) \in \mathbb{C}^n \mid \cdots \operatorname{Im} \zeta_i < \operatorname{Im} \zeta_{i+1} < \cdots, \\ \sum_{i=1}^{n-1} (\operatorname{Im} \zeta_{i+1} - \operatorname{Im} \zeta_i) < \beta \Big\} \end{aligned}$$

and continuous on $\bar{T}_n^\beta$. The restrictions of the analytically continued Green functions to the so-called Euclidean region

$$E_n^\beta \equiv \{ \mathbf{z} \in C^n \mid \operatorname{Re} z_i = 0;\ -\beta/2 \leqslant \operatorname{Im} z_1 \leqslant \cdots \leqslant \operatorname{Im} z_i \leqslant \operatorname{Im} z_{i+1} \leqslant \cdots \leqslant \beta/2 \}$$

will be called Euclidean Green functions of the free Bose gas and their full collection extended to $\bigcup_{n\geqslant 0}\mathscr{W}(\mathbf{h})^{\times n}$ by linearity will be denoted by ${}^{E}\mathbb{G}^{0}$. The following abbreviations will be used:

$$E_n^{\beta,+}=\{(S_1,\ldots,S_n)\in E_n^\beta \mid 0\leqslant S_i\} \tag{2.11}$$

$$\mathbf{S}^k\equiv(S_1^k,\ldots,S_k^k)\in E_k^\beta \tag{2.12}$$

$$\mathbf{W}^k\equiv(W_1^k,\ldots,W_k^k)\in\mathscr{W}(\mathbf{h})^{\times k} \tag{2.13}$$

$${}^{E}G^0_{\mathbf{W}^k}(\mathbf{S}^k)\equiv {}^{E}G^0_{W_1,\ldots,W_k}(S_1,\ldots,S_k) \tag{2.14}$$

$$\mathbf{f}^k\equiv(f_1,\ldots,f_k)\in L_2(\mathbb{R}^d)^{\times k} \tag{2.15}$$

$$\begin{aligned}{}^{E}G^0_{\mathbf{f}^k}(\mathbf{S}^k)&\equiv {}^{E}G^0_{(f_1,\ldots,f_k)}(S_1,\ldots,S_k)\\ &= {}^{E}G^0_{(W(f_1),\ldots,W(f_k))}(S_1,\ldots,S_k)\end{aligned} \tag{2.16}$$

$$\mathbf{S}^{k*}\equiv(-S_k,\ldots,-S_1)\qquad\text{for}\quad \mathbf{S}^k\in E_k^\beta \tag{2.17}$$

$$\mathbf{W}^{k*}\equiv(W_k^+,\ldots,W_1^+)\qquad\text{for}\quad \mathbf{W}^k=(W_1,\ldots,W_k) \tag{2.18}$$

$$(\mathbf{W}^n,\mathbf{S}^n)\equiv((W_1,S_1),\ldots,(W_n,S_n)) \tag{2.19}$$

Proposition 2.2. Let

$${}^{E}\mathbb{G}^0=\{{}^{E}G_{W_1,\ldots,W_k}(S_1,\ldots,S_k)\mid W_i\in\mathscr{W}(\mathbf{h}),\,(S_1,\ldots,S_k)\in E_k^\beta\}$$

be the collection of the Euclidean Green functions of the free Bose gas in the noncritical regime. Then the collection ${}^{E}\mathbb{G}^0$ has the following properties:

EG(1) (i) For each fixed $\mathbf{W}^k\in\mathscr{W}(\mathbf{h})^{\times k}$ the map

$$E_n^\beta\ni\mathbf{S}^k\to{}^{E}G^0_{\mathbf{W}^k}(\mathbf{S}^k)$$

is continuous.

(ii) For each fixed $\mathbf{S}^k\in E_k^\beta$ the map

$$\mathscr{W}(\mathbf{h})^{\times k}\ni\mathbf{W}^k\to{}^{E}G^0_{\mathbf{W}^k}(\mathbf{S}^k)$$

is multilinear and for any $\mathbf{f}^k\in L_2(\mathbb{R}^d)^{\times k}$ the map

$$L_2(\mathbb{R}^d)^{\times k}\ni\mathbf{f}^k\to{}^{E}G^0_{\mathbf{f}^k}(\mathbf{S}^k)$$

is continuous and obeys the estimate $|{}^{E}G_{\mathbf{f}^k}(\mathbf{S}^k)|\leqslant 1$.

(iii) For any $\mathbf{S}^k \in E_k^\beta$ and any $S \in [-\beta/2, \beta/2]$ such that $S_k + S \leqslant \beta/2$ the Euclidean Green functions are locally shift invariant, i.e., for any $\mathbf{W}^k \in \mathscr{W}(\mathbf{h})^{\times k}$

$$^E G^0_{\mathbf{W}_k}(\mathbf{S}^k + S) = {}^E G^0_{\mathbf{W}^k}(\mathbf{S}^k)$$

where $\mathbf{S}^k + S \equiv (S_1 + S, ..., S_k + S)$.

(iv) For any $\mathbf{W}^k \in \mathscr{W}(\mathbf{h})^{\times k}$, any $\mathbf{S}^k$: $\exists_{1 \leqslant i \leqslant k-1}\ S_i = S_{i+1}$ we have the equality

$$^E G^0_{\mathbf{W}^k}(\mathbf{S}^k) = {}^E G^0_{\mathbf{W}^k_{(i)}}(\mathbf{S}^k_{(i)})$$

where

$$\mathbf{W}^k_{(i)} = (W_1, ..., W_{i-1}, W_i \cdot W_{i+1}, ..., W_k)$$
$$\mathbf{S}^k_{(i)} = (S_1, ..., S_i, S_{i+2}, ..., S_k)$$

(v) For any $\mathbf{W}^k \in \mathscr{W}(\mathbf{h})^{\times k}$: $\exists_{1 \leqslant i \leqslant k}$: $W_i = \mathbf{1}$ the following equality holds

$$^E G^0_{\mathbf{W}^k}(\mathbf{S}^k) = {}^E G^0_{{}_{(i)}\mathbf{W}^{k-1}}({}_{(i)}\mathbf{S}^{k-1})$$

where

$${}_{(i)}\mathbf{W}^{(k-1)} = (W_1, ..., W_{i-1}, W_{i+1}, ..., W_k)$$
$${}_{(i)}\mathbf{S}^{(k-1)} \equiv (S_1, ..., S_{i-1}, S_{i+1}, ..., S_k)$$

(vi) $^E G^0_{\mathbf{1}}(0) = 1$.

EG(2) (OS-positivity). For any terminating sequences

$$\underset{\sim}{\mathbf{W}} = (\mathbf{W}^0, \mathbf{W}^1, ..., \mathbf{W}^k, ...), \qquad \underset{\sim}{\mathbf{S}} = (\mathbf{S}^0, ..., \mathbf{S}^k, ...)$$

with

$$\begin{gathered} \mathbf{S}^k \in E_k^{\beta,+} \qquad \text{for all} \quad k = 1, 2, ... \\ \sum_{k,l} {}^E G^0_{\mathbf{W}^{k,*}, \mathbf{W}^l}(\mathbf{S}^{k,*}, \mathbf{S}^l) \geqslant 0 \end{gathered} \tag{2.20}$$

EG(3) For any terminating sequences

$$\underset{\sim}{\mathbf{W}} = (\mathbf{W}^0, \mathbf{W}^1, ..., \mathbf{W}^k, ...), \qquad \underset{\sim}{\mathbf{S}} = (\mathbf{S}^0, ..., \mathbf{S}^k, ...)$$

with

$$\underset{\sim}{\mathbf{S}}^k \in E_k^{\beta,+} \qquad \text{for all} \quad k = 1, 2, ...$$

and for any $f \in L_2(\mathbb{R}^d)$

$$\sum_{k,l} {}^E G^0_{\underset{\sim}{\mathbf{W}}^{k,*}, \bar{f}; \bar{f}, W^l}(\mathbf{S}^{k,*}, 0, 0, S^l) \leqslant \sum_{k,l} {}^E G^0_{\underset{\sim}{\mathbf{W}}^k, \underset{\sim}{\mathbf{W}}^l}(\underset{\sim}{\mathbf{S}}^{k,*}, \underset{\sim}{\mathbf{S}}^l) \tag{2.21}$$

EG(4) (Weak form of the KMS condition). Let

$${}^E\hat{G}^0_{W_0,\dots,W_n}(S_1,\dots,S_n) \equiv {}^E G^0_{W_0, W_1,\dots,W_n}\left(-\frac{\beta}{2}, S_1 - \frac{\beta}{2},\dots, S_n - \frac{\beta}{2}\right)$$

for $0 \leqslant S_1 \leqslant \cdots \leqslant S_n \leqslant \beta$. Then for each n, $\mathbf{W}^{n+1} \in \mathscr{W}(\mathbf{h})^{\times n}$,

$${}^E\hat{G}^0_{\mathbf{W}^{n+1}}(\mathbf{S}^n) = {}^E\hat{G}^0_{W_n, W_0,\dots,W_{n-1}}(\beta - S_n, \beta - S_n + S_1,\dots, \beta - S_n + S_{n-1}) \tag{2.22}$$

EG(5) (Euclidean invariance and uniqueness of the vacuum). Under the natural action $\tau_{\{a,A\}}$ of the Euclidean group of motions $E(d)$ on the Weyl algebra $W(\mathbf{h})$ the Euclidean Green functions are:

(i) Invariant.

(ii) Have the cluster decomposition property, i.e., for any

$$\mathbf{W}^k \in \mathscr{W}(\mathbf{h})^{\times k}, \qquad \mathbf{W}^l \in \mathscr{W}(\mathbf{h})^{\times l}, \qquad \mathbf{S}^k \in E^\beta_k, \qquad \mathbf{S}^l \in E^\beta_l$$

we have

$$\lim_{|a| \to \infty} {}^E G^0_{\tau_{\{a,0\}}\mathbf{W}^k; \mathbf{W}^l}(\mathbf{S}^k, \mathbf{S}^l) = {}^E G^0_{\mathbf{W}^k}(\mathbf{S}^k) \cdot {}^E G_{\mathbf{W}^l}(\mathbf{S}^l) \tag{2.23}$$

Proof. Let us consider the free gas GNS W^*-KMS structure $\mathbb{C}_0 = (\mathscr{H}_0, \Omega_0, \alpha^0_t, \pi_0(\mathscr{W}(\mathbf{h}))'')$. By the Araki theorem[31] the Euclidean Green functions are represented as

$${}^E G^0_{\mathbf{W}^n}(\mathbf{S}^n) = \langle \Omega_0 | \alpha^0_{is_1}(\pi_0((W_1)) \cdots \alpha^0_{is_n}(\pi_0(W_n)) \Omega_0 \rangle \tag{2.24}$$

and by the very definition of $\mathbb{C}_0$

$$\omega_0^{(\beta,\mu)}(W(f)) = \langle \Omega_0, \pi_0(W(f)) \Omega_0 \rangle \tag{2.25}$$

Now everything follows easily from (2.24) and the Araki theorem. In particular the OS positivity EG(2) follows from the fact that the l.h.s. of (2.20) can be written as

$$\text{l.h.s. of (2.20)} \equiv \left\langle \left(\sum_k \prod_{l_k=1}^{k} \alpha^0_{is^k_{l_k}}(\pi_0(W(f^k_{l_k}))) \right) |\Omega_0| \right.$$
$$\left. \left(\sum_k \prod_{l_k=1}^{k} \alpha^0_{is^k_{l_k}}(\pi_0(W(f^k_{i_k}))) \Omega_0 \right) \right\rangle \tag{2.26}$$

The weak form of the KMS condition, formulated as EG(4), can be observed easily from the explicit formula (2.8) for the corresponding Green functions. ▮

Remarks. As demonstrated in ref. 20, the multitime Euclidean Green functions of any C^*- (or W^*-) KMS structure obey similar properties EG(1)–EG(4) with the obvious modifications of the continuity properties EG(2)(ii) and EG(3). It can be checked using the basic results of refs. 1–4 that the Euclidean Green functions of dilute Bose gases (and also of dilute Fermi gases built over the CAR algebra over $\mathbf{h}$) in the regime considered by Ginibre[1] obey the system EG(1)–EG(5). The detailed study of the modular structures that arise (see below) are now under investigation. The Euclidean Green functions of the critical Bose gas also obey properties similar to EG(1)–EG(5i) and their restrictions to the Abelian sector (of the Weyl algebra) fulfill also EG(6) (see below).

The complex subalgebra $\mathscr{A}(\mathbf{h})$ of $\mathscr{W}(\mathbf{h})$ generated by the elements $W(f)$ with $f=\bar{f}$ will be called an Abelian sector of $\mathscr{W}(\mathbf{h})$ and the corresponding free Euclidean Green functions restricted to $\mathscr{A}(\mathbf{h})$ will be denoted by ${}^{EA}\mathbb{G}_0$. For $-\beta/2 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant \beta/2$ we have

$$ {}^{EA}G_0((s_1, f_1),..., (s_n, f_n)) = \prod_{1 \leqslant i \leqslant j \leqslant n} \exp -\tfrac{1}{2} S_0^\beta(s_j - s_i, f_i \otimes f_n) \quad (2.27) $$

where

$$ S_0^\beta(s, f_i \otimes f_n) = \int \hat{S}_0^\beta(s, p)\, \overline{f_i(p)}\, f_j(p)\, dp \quad (2.28) $$

$$ \hat{S}_0^\beta(s, p) \equiv \frac{z^{s/\beta} e^{-s\mathscr{E}(p)} + z^{1-s/\beta} e^{-(\beta - s)\mathscr{E}(p)}}{1 - z e^{-\beta\mathscr{E}(p)}} \quad (2.29) $$

The periodic extension of $\hat{S}_0^\beta(s, p)$ to the whole $\mathbb{R}$ shall be denoted by the same symbol. The fundamental properties of the free thermal kernels $S_0^\beta(s, x)$ are collected in the following proposition.

Proposition 2.3. 1. Let $\hat{S}_0^\beta$ be the free thermal kernel (2.29) with $\mathscr{E}(p) = p^2$ or $\mathscr{E}(p) = (p^2 + m^2)^{1/2}$, $m > 0$. Then for any $0 \leqslant s \leqslant \beta$, and z noncritical, we have:

(i) $0 < S_0^\beta(s, \cdot) \in S(\mathbb{R}^d)$ if $s \in (0, \beta)$.

(ii) $S_0^\beta(0, \cdot) = S_0^\beta(\beta, \cdot) = C_0^\beta(\cdot)$ in $\mathscr{D}'(\mathbb{R}^d)$ sense.

2. Let $\mathscr{E}(p) = |p|$; then for any $0 \leqslant s \leqslant \beta$, $0 < z < 1$, we have:

(i) $0 < S_0^\beta(s, \cdot) \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ if $s \in (0, \beta)$.

(ii) $S_0^\beta(0, \cdot) = S_0^\beta(\beta, \cdot) = C_0^\beta(\cdot)$ in $\mathscr{D}'(\mathbb{R}^d)$ sense.

3. For $\mathscr{E}(p)=p^2$ or $\mathscr{E}(p)=(p^2+m^2)^{1/2}$, $m\geqslant 0$, the kernel S_0^β is stochastically positive on the space $L_2(K_\beta\times\mathbb{R}^d)$, i.e., for any $g_1, g_2\in L_2(K_\beta)$; $f_1,..., f_n\in L_2(\mathbb{R}^d)$, $C_1,..., C_n\in C$, we have

$$\sum_{\alpha,\beta=1}^{n} C_\alpha \overline{C_\beta} S_0^\beta(g_\alpha\otimes f_\alpha \mid g_\beta\otimes f_\beta)\geqslant 0 \tag{2.30}$$

where

$$S_0^\beta(g\otimes f\mid g'\otimes f')=\int_0^\beta ds\int_0^\beta ds'\,\overline{g(s)}\,g'(s')\int dx\int dy \\ \times\overline{f(x)}\,f'(y)\,S_0^\beta(|s-s'|, x-y) \tag{2.31}$$

4. For $\mathscr{E}(p)=p^2$ or $\mathscr{E}(p)=(p^2+m^2)^{1/2}$, $m\geqslant 0$, the kernel S_0^β is OS positive on the circle K_β, i.e., for any $t_1,..., t_n$ in $[0, \beta/2]$, $f_1,..., f_n\in L_2^r(\mathbb{R}^d)$, $c_1,..., c_n\in C$, we have

$$\sum_{\alpha,\beta}\bar{c}_\alpha c_\beta\int dx\int dy\, S_0^\beta(t_\alpha+t_\beta\mid f_\alpha\otimes f_\beta)\geqslant 0 \tag{2.32}$$

Proof. From the assumption $\sup_p ze^{-\beta\mathscr{E}(p)}<1$ it follows that

$$\hat{S}_0^\beta(s, p)=\sum_{n\geqslant 0}\widehat{F_n}(s, p) \tag{2.33}$$

where

$$\hat{F}_n(s, p)\equiv z^{n+s/\beta}e^{-(\beta n+s)\mathscr{E}(p)}+z^{n+1-s/\beta}e^{-(\beta(n+1)-s)\mathscr{E}(p)} \tag{2.34}$$

So, if $\mathscr{E}(p)=p^2$ or $\mathscr{E}(p)=(p^2+m^2)^{1/2}$, $m>0$, then $\hat{F}_n(s, p)\in S(\mathbb{R}^d)$ for each $n\geqslant 1$ and $n=0$ if $s\in(0,\beta)$. In this case also $\sum_{n\geqslant 1}^{\infty} z^n\exp[-\beta n\mathscr{E}(p)]\in S(\mathbb{R}^d)$. Taking into account that

$$\widehat{S_0^\beta}(s, p)=\left(\sum_{n\geqslant 0} z^n e^{-\beta n\mathscr{E}(p)}\right)(z^{s/\beta}e^{-s\mathscr{E}(p)}+z^{1-s/\beta}e^{-(\beta-s)\mathscr{E}(p)})$$

it follows that also $\widehat{S_0^\beta}(s, p)\in S(\mathbb{R}^d)$ if $s\in(0,\beta)$. Moreover, $S_0^\beta(s, x)>0$ for any $x\in\mathbb{R}^d$.

Similarly, if $\mathscr{E}(p)=|p|$, then we have

$$\hat{S}_0^\beta(s, p)=\left(\sum_{n\geqslant 0} z^n e^{-\beta n|p|}\right)(z^{s/\beta}e^{-s|p|}+z^{1-s/\beta}e^{-(\beta-s)|p|}) \tag{2.35}$$

Therefore from the continuity of the Fourier transform and (2.4) we obtain

$$\begin{aligned} S_0^\beta(s,x) &= \sum_{n\geq 0} \frac{z^n c}{(\beta^2 n^2 + |x|^2)^{d+1/2}} \\ &\quad\times\left(\frac{z^{s/\beta}c}{(s^2+|x|^2)^{d+1/2}} + \frac{z^{1-s/\beta}c}{(\beta-s)^2+|x|^2)^{d+1/2}}\right) \\ &= \sum_{n\geq 0} z^{n+s/\beta}\frac{c}{[(\beta n+s)^2+|x|^2]^{d+1/2}} \\ &\quad+\sum_{n\geq 0} z^{n+1-s/\beta}\frac{c}{\{[\beta(n+1)+s]^2+|x|^2\}^{d+1/2}} \end{aligned} \tag{2.36}$$

The above series are uniformly convergent on $\mathbb{R}^d$ and define a continuous function with decay at least as $1/(s^2+|x|^2)^{d+1/2}$ for $|x|\uparrow\infty$, which is integrable provided $s>0$.

Although claims 3 and 4 follow easily from a basic characterization theorem of KL[32] we present simple proofs for the reader's convenience. Expanding into the Fourier series the periodic function $\hat{S}^\beta(s,p)$ we obtain:

$$\begin{aligned} \hat{S}^\beta(s,p) &= \sum_{n\in\mathbb{Z}} \{[\beta(\mu+\mathscr{E}(p))]^2+(2\pi n)^2\}^{-1}\, 2\beta[\mu\beta+\mathscr{E}(p)] \\ &\quad\times(1-e^{-\beta[\mu+\mathscr{E}(p)]})(1-ze^{-\beta\mathscr{E}(p)})^{-1}\, e^{i2\pi ns/\beta} \end{aligned} \tag{2.37}$$

Because all the Fourier coefficients in the expansion (2.37) are positive, the stochastic positivity (2.30) follows. The OS positivity of the one-time Euclidean Green function is a general feature of all KMS systems, as demonstrated in ref. 19. The straightforward proof of 4 is as follows. Let

$$\mathbb{C}_0 = (\mathscr{H}_0, \Omega_0, \pi_0, \alpha_t^0; \pi_0(\mathscr{W}(h))'')$$

be the basic GNS W^*-KMS system of the free Bose gas. Then we can write

$$\begin{aligned} \sum_{\alpha,\beta} c_\alpha c_\beta S_0^\beta(s_\alpha+s_\beta\,|\,\bar{f}_\alpha\otimes f_\beta) &= \left\|\sum_\alpha c_\alpha \alpha^0_{is_\alpha}(\pi_0(\mathscr{W}(f_\alpha))\,\Omega_0\right\|^2 \\ &\geq 0 \quad \blacksquare \end{aligned} \tag{2.38}$$

Remarks. 1. Let h_0^μ be a nonnegative, self-adjoint generator of unitary group $U_t^0 = z^{-it/\beta}e^{-it\mathscr{E}(p)}$ acting in the space $\mathbf{h}=L_2(\mathbb{R}^d)$ and let dP^μ be the corresponding spectral measure of h_0^μ. Then, defining the covariance operator

$$\Gamma_0^\beta(s) \equiv \int_0^\infty \frac{dP^\mu(\lambda)}{1-e^{-\beta\lambda}}\,(e^{-s\lambda}+e^{-(\beta-s)\lambda}) \tag{2.39}$$

acting in $\mathbf{h}$ by definition

$$\langle f, \Gamma_0^\beta g\rangle \equiv \hat{S}_0^\beta(s \mid f \otimes g) \tag{2.40}$$

we see that the kernel $S_0^\beta(s \mid f \otimes g)$ belongs to the class of kernels considered in ref. 32.

2. Let us observe that the periodic kernels ${}^nS_\beta(s, p) \equiv F_n(s, p)$ for each n also have the positivity properties stated in points 3 and 4 of Proposition 2.3. This leads to an interesting decomposition of the free thermal process ξ_t^0 defined below as a sum of independent OS-positive Gaussian processes $\xi_t^{0, n}$, which have covariances equal to ${}^nS_\beta(s, p)$. This decomposition might be eventually used to develop a rigorous renormalisation group analysis of interacting Boses gases.

Proposition 2.4. The collection ${}^{EA}\mathbb{G}_0$ of the Euclidean Green functions of the free Bose gas in the noncritical regime obeys the properties EG(1)–EG(5) of Proposition 2.2 and additionally:

EG(6) (Stochastic positivity). For any

$$\underset{\sim}{S}^k \in E_k^\beta, \qquad f^k = (f_1^k, \ldots, f_k^k): f_i^k = \bar{f}_i^k \in L_2(\mathbb{R}^d)$$

we have

$$\sum_{k, l} {}^E G^0_{\underset{\sim}{f}^{k*}, \underset{\sim}{f}^l}(\underset{\sim}{s}^k, \underset{\sim}{s}^l) \geqslant 0 \tag{2.41}$$

Proof. From assertion 3 of Proposition 2.3 it follows by standard construction (see, e.g., refs. 28 and 32) that there exists a Gaussian process $(\xi_t^0)_{t \in K_\beta}$ indexed by $L^2(\mathbb{R}^d)$ with mean zero and the covariance given by $S_0^\beta(\tau, x)$. The r.h.s. of (2.41) can be rewritten in terms of (ξ_t^0) as

$$E\left|\sum_k \prod_{l_k=1}^{n_k} \exp i\langle \xi^0_{s^k_{l_k}}; f^k_{l_k}\rangle\right|^2 \qquad \blacksquare$$

Having defined a system of Euclidean multitime Green functions with the properties listed in Proposition 2.2, we can apply the constructions of ref. 20 to build certain W^*-KMS structures. The interesting aspect of the proposition below is that the system of Euclidean Green functions of the free Bose gas restricted to $\mathscr{A}(\mathbf{h})$ already contains all information of the free Bose gas.

Proposition 2.5. Let $\mathscr{E}(p) = p^2$ or $\mathscr{E}(p) = (p^2 + m^2)^{1/2}$, $m \geqslant 0$, and let z be noncritical. Then:

1. There exists a unique (up to a unitary equivalence) W^*-KMS system ${}^E\mathbb{C}=({}^E\mathscr{H}_0, {}^E\Omega_0, {}^E\alpha^0_t, {}^E\mathbf{m}_0)$ and a bounded $*$-representation ${}^E\pi_0$ of $\mathscr{W}(\mathbf{h})$ such that:

(i) ${}^E\pi_0(\mathscr{W}(\mathbf{h}))\subseteq {}^E\mathbf{m}_0$.

(ii) The multitime Euclidean Green functions of ${}^E\mathbb{C}_0$ restricted to ${}^E\pi_0(\mathscr{W}(\mathbf{h}))$ coincide with ${}^E\mathbb{G}_0$.

(iii) We have

$$ {}^E\mathbf{m}_0 = W^*\{{}^E\alpha^0_{t_1}({}^E\pi_0(W(f_1)))\cdots {}^E\alpha^0_{t_n}({}^E\pi_0(W(f_n)))\} $$

2. There exists a unique (up to a unitary equivalence) W^*-KMS system ${}^A\mathbb{C}=({}^A\mathscr{H}_0, {}^A\Omega_0, {}^A\alpha^0_t, {}^A\mathbf{m}_0)$ and a bounded $*$-representation ${}^A\pi_0$ of $\mathscr{A}(\mathbf{h})$ such that:

(i) ${}^A\pi_0(\mathscr{A}(\mathbf{h}))\subseteq {}^A\mathbf{m}_0$.

(ii) The multitime Euclidean Green functions of the system ${}^A\mathbb{C}_0$ restricted to ${}^A\pi_0(\mathscr{A}(\mathbf{h}))$ coincide with ${}^A\mathbb{G}_0$.

(iii) We have

$$ {}^A\mathbf{m}_0 = W^*\{{}^A\alpha^0_{t_1}({}^A\pi_0(W_1))\cdots {}^A\alpha^0_{t_n}({}^A\pi_0(W_n))\} $$

for $W_1,..., W_n\in\mathscr{A}(\mathbf{h})$.

3. Both systems ${}^E\mathbb{C}_0$ and ${}^A\mathbb{C}_0$ are unitarily equivalent to the GNS W^*-KMS system $\mathbb{C}_0=(\mathscr{H}_0, \Omega_0, \alpha^0_t, \pi_0(\mathscr{W}(\mathbf{h}))'')$.

Proof. Step 1. In the first step we apply in a sketchy way a general construction of ref. 20 (see also ref. 19), to which we refer for more details. Because in both cases the constructions of ${}^E\mathbb{C}_0$ and ${}^A\mathbb{C}_0$ are identical, we restrict ourselves to the construction of ${}^E\mathbb{C}_0$ only.

Let $\tilde{V}^\beta$ be the free complex vector space built over the set $\{(\underset{\sim}{W}^n, \underset{\sim}{s}^n)\mid \underset{\sim}{s}^n\in E^{\beta,+}_n\}$. Then we divide $\tilde{V}^\beta$ by the natural relations arising from the properties EG(1)(i), EG(1)(iv), EG(1)(v), and EG(1)(vi), obtaining a complex vector space V^β. The sesquilinear form

$$ \left(\sum_\alpha c_\alpha(\underset{\sim}{W}^{n_\alpha}, \underset{\sim}{s}^{n_\alpha}); \sum_\beta d_\beta(\underset{\sim}{W}^{k_\beta}, \underset{\sim}{s}^{k_\beta})\right) \equiv \sum_{\alpha,\beta} \bar{c}_\alpha\, d_\beta\, {}^E G^0_{\underset{\sim}{W}^{n_\alpha *},\, \underset{\sim}{W}^{k_\beta}}(\underset{\sim}{s}^{n_\alpha *}, \underset{\sim}{s}^{k_\beta}) \tag{2.42} $$

defined on V^β is nonnegative by EG(2). The corresponding Hilbert space will be denoted by ${}^E\mathscr{H}_0$ and the corresponding classes of abstraction will be denoted by square brackets $[\cdot]$.

Lifting the natural action ${}^E\tilde{\pi}_0$ of $\mathscr{W}(\mathbf{h})$ on $\tilde{V}^\beta$, defined by

$$ {}^E\tilde{\pi}_0(W)(\underset{\sim}{W}^n, \underset{\sim}{s}^n) \equiv ((W, \underset{\sim}{W}^n); (0, \underset{\sim}{s}^n)) $$

to the space ${}^E\mathscr{H}_0$, we obtain a *-representation of $\mathscr{W}(\mathbf{h})$ in ${}^E\mathscr{H}_0$ which is bounded because of EG(3).

Lifting the local shift transformation given by EG(1)(iii) into the space ${}^E\mathscr{H}_0$, we obtain a uniquely determined self-adjoint generator ${}^EH_0$. Defining ${}^E\Omega_0 = [(1, 0)] \in {}^E\mathscr{H}_0$, we have that for any $[(\underset{\sim}{W}^n, \underset{\sim}{s}^n)] \in {}^E\mathscr{H}_0$,

$$ {}^E\alpha^0_{is_n}({}^E\pi_0(W_1)) \cdots {}^E\alpha^0_{is_1}({}^E\pi_0(W_n))\, {}^E\Omega_0 = [(\underset{\sim}{W}^n, \underset{\sim}{s}^n)] $$

Moreover, the vector-valued maps

$$ E^{\beta,+}_n \ni \underset{\sim}{s}^n \to \prod_{k=1}^{n} {}^E\alpha^0_{is_k}({}^E\pi_0(W_k))\, {}^E\Omega \in {}^E\mathscr{H}_0 $$

can be holomorphically extended to the tube T^n_β being continuous on the boundary ∂T^n_β. In particular, it can be proved[20] that the vector ${}^E\Omega_0$ is cyclic and separating for the W^*-closure ${}^E\mathbf{m}_0$ of the *-algebra generated by all products:

$$ {}^E\alpha^0_{t_1}({}^E\pi_0(W_1)) \cdots {}^E\alpha^0_{t_n}(W_n)) $$

where $t_1,..., t_n \in \mathbb{R}$; $W_1,..., W_n \in \mathscr{W}(\mathbf{h})$. Thus we have sketched the construction and the proof that ${}^E\mathbb{C}_0 \equiv ({}^E\mathscr{H}_0, {}^E\Omega_0; {}^E\alpha^0_t; {}^E\mathbf{m}_0)$ forms a W^*-KMS system. The Euclidean Green functions of the system ${}^E\mathbb{C}_0$ are equal to ${}^E\mathbb{G}_0$ by the very construction. Let ${}^E\mathbb{C}'_0 \equiv ({}^E\mathscr{H}'_0, {}^E\Omega'_0; {}^E\alpha^{0\prime}_t; {}^E\mathbf{m}'_0)$ be another W^*-KMS system whose Euclidean Green function coincides with ${}^E\mathbb{G}_0$ and such that ${}^E\mathbf{m}_0 \supset {}^E\pi'_0(\mathscr{W}(\mathbf{h}))$ for some

$$ {}^E\pi'_0 \in \mathrm{Rep}^*(\mathscr{W}(\mathbf{h}), L({}^E\mathscr{H}'_0)) $$

$$ {}^E\mathbf{m}'_0 = W^*\{{}^E\alpha^{0\prime}_{t_1}({}^E\pi'_0(W_1)) \cdots {}^E\alpha^{0\prime}_{t_n}({}^E\pi'_0(W_n))\} $$

Then the isometry

$$ j\colon {}^E\alpha^0_{t_1}({}^E\pi_0(W_1)) \cdots {}^E\alpha^0_{t_n}({}^E\pi_0(W_n))\, {}^E\Omega_0 $$

$$ {}^E\alpha^{0\prime}_{t_1}({}^E\pi'_0(W_1)) \cdots {}^E\alpha^{0\prime}_{t_n}({}^E\pi'_0(W_n))\, {}^E\Omega'_0 $$

can be extended to a unitary operator such that $j\,{}^E\Omega_0 = {}^E\Omega'_0$; ${}^E\alpha^0_t = j^{-1}\, {}^E\alpha^{0\prime}_t j$; ${}^E\mathbf{m}'_0 = j\, {}^E\mathbf{m}_0 j^{-1}$.

Step 2. In the second step we identify the W^*-KMS system ${}^E\mathbb{C}_0$ with $\mathbb{C}_0$. Although this identification follows from Section V of ref. 20, we present a straightforward proof below. To start with, let us define a linear space $\mathscr{D}^E$ generated by

$$\{{}^E\alpha^0_{is_n}({}^E\pi_0(W(f_n)))\cdots{}^E\alpha^0_{is_1}({}^E\pi_0(W(f_1)))\,{}^E\Omega_0 \,|\, \underset{\sim}{s}^n\in E^{\beta,+}_n,\ \underset{\sim}{f}^n\in L^2(\mathbb{R}^d)^{\otimes n}\}$$

From step 1 we know that $\overline{\mathscr{D}^E}={}^E\mathscr{H}_0$ and for any $\underset{\sim}{f}^n\in L_2(\mathbb{R}^d)^{\times n}$ the map

$$E^{\beta,+}_n\ni\underset{\sim}{s}^n\to T-\prod_{i=1}^n{}^E\alpha^0_{is_i}(W(f_i))\,{}^E\Omega_0$$

can be uniquely extended to a holomorphic, vector-valued function on the tube T^β_n and this extension gives also the holomorphic extension of the corresponding Green function.

Computing the r.h.s. of

$$\begin{aligned}&\langle{}^E\alpha^0_{i\tau_n}({}^E\pi_0(W(f_1)))\cdots{}^E\alpha^0_{i\tau_1}({}^E\pi_0(W(f_n)))\,{}^E\Omega_0,\\ &\quad{}^E\alpha^0_t(W(f))\,{}^E\alpha^0_{is_m}({}^E\pi_0(W(g_1)))\cdots{}^E\alpha^0_{is_1}({}^E\pi_0(W(g_m)))\,{}^E\Omega_0\rangle\\ &=G^0((\bar f_n,-i\tau_n),\ldots,(\bar f_1,-i\tau_1);(f,t),(g_1,is_1),\ldots,(g_m,is_m))\end{aligned}\tag{2.43}$$

with the help of the formula (2.8) and comparing it with

$$\begin{aligned}&\langle{}^E\alpha^0_{i\tau_n}({}^E\pi_0(W(f_n)))\cdots{}^E\alpha^0_{i\tau_1}({}^E\pi_0(W(f_1)))\,{}^E\Omega_0;\\ &\quad W[z^{-it/\beta}e^{it\mathscr{E}(p)}f]\cdot{}^E\alpha^0_{is_n}({}^E\pi_0(W(g_n)))\cdots{}^E\alpha^0_{is_1}({}^E\pi_0(W(g_1)))\,{}^E\Omega_0\rangle\end{aligned}\tag{2.44}$$

we conclude that

$${}^E\alpha^0_t({}^E\pi_0(W(f)))={}^E\pi_0(W(z^{it/\beta}e^{it\mathscr{E}}f))\tag{2.45}$$

on a dense domain $\mathscr{D}^E$ and thus on ${}^E\mathscr{H}_0$.

Defining a map

$$\begin{aligned}j_E\colon&{}^E\alpha^0_{is_n}({}^E\pi_0(W(f_n)))\cdots{}^E\alpha^0_{is_1}({}^E\pi_0(W(f_1)))\,{}^E\Omega_0\\ &\to\alpha^0_{is_n}(\pi_0(W(f_n)))\cdots\alpha^0_{is_1}(\pi_0(W(f_n)))\,\Omega_0\in\mathscr{H}_0\end{aligned}\tag{2.46}$$

we obtain a densely defined map with a dense range isometry from ${}^E\mathscr{H}_0$ to $\mathscr{H}_0$ which extends naturally to a unitary map j_E. From (2.45) we have

$$j_E\,{}^E\alpha^0_t\,j_E^{-1}=\alpha^0_t,\qquad j_E\,{}^E\Omega_0=\Omega_0,\qquad {}^E\mathbf{m}_0=j_E^{-1}\pi_0(W(h))''\,j_E$$

Step 3. In the third step we identify the W^*-KMS system ${}^A\mathbb{C}_0$ with $\mathbb{C}_0$. The following lemma, whose proof is translated into the fourth step below, plays a basic role.

Lemma 2.6. Let $\mathscr{E}(p)=p^2$ or $\mathscr{E}(p)=(p^2+m^2)^{1/2}$, $m\geqslant 0$. Then the set of functions

$$V=\{e^{it\mathscr{E}(p)}f(p)\mid t\in\mathbb{R};\ f=\bar{f}\in L_2(\mathbb{R}^d)\}$$

is $\mathbb{R}$-linearly dense in $L^2(\mathbb{R}^d)$.

It is because the Euclidean Green functions restricted to the Abelian sector $\mathscr{A}(\mathbf{h})$ of $\mathscr{W}(\mathbf{h})$ obey the properties EG(1)–EG(4) that we can apply the construction presented in step 1 obtaining again a W^*-KMS system ${}^A\mathbb{C}_0=({}^A\mathscr{H}_0, {}^A\Omega_0, {}^A\alpha_t^0; {}^A\mathbf{m}_0)$, where ${}^A\mathbf{m}_0$ is the W^*-algebra generated by the operators

$${}^A\alpha_{t_1}^0({}^A\pi_0(W(f_1)))\cdots {}^A\alpha_{t_n}^0({}^A\pi_0(W(f_n)))$$

where ${}^A\pi_0$ is the corresponding representation of $\mathscr{A}(\mathbf{h})$ in $L({}^A\mathscr{H}_0)$ and all f_i are real. From the cyclicity of ${}^A\Omega_0$ under the action of ${}^A\mathbf{m}_0$ it follows that the set of vectors

$${}^A\alpha_{t_1}^0({}^A\pi_0(W(f_1)))\cdots {}^A\alpha_{t_n}^0({}^A\pi_0(W(f_n)))\ {}^A\Omega_0$$

is linearly dense in ${}^A\mathscr{H}_0$. Defining a map

$$\begin{aligned} j_A:\ &{}^A\alpha_{t_1}^0({}^A\pi_0(W(f_1)))\cdots {}^A\alpha_{t_n}^0({}^A\pi_0(W(f_n)))\ {}^A\Omega_0 \\ &\to \alpha_{t_1}^0(\pi_0(W(f_1)))\cdots \alpha_{t_n}^0(\pi_0(W(f_n)))\ \Omega_0 \\ &\equiv \pi_0\left(W\left(\sum_{\alpha=1}^{n} e^{it_\alpha h^\mu} f_\alpha\right)\right)\Omega_0 \\ &\quad\times \prod_{1\leqslant\alpha<\beta\leqslant n} \exp\{-i\sigma(e^{it_\alpha h^\mu}f_\alpha;\ e^{it_\beta h^\mu}f_\beta)\} \end{aligned} \tag{2.47}$$

we see that it is an isometry with dense range because of Lemma 2.6. Moreover, $j_A({}^A\Omega_0)=\Omega_0$.

Computing

$$\begin{aligned} &\left(j_A \sum_{l=1}^{n} {}^A\alpha_{it_l}^0({}^A\pi_0(W(g_l)))\, j_A^*\right) \prod_{k=1}^{m} \pi_0(W(e^{-i(S_k h^\mu)}f_k))\,\Omega_0 \\ &= \prod_{l=1}^{n} \alpha_{it_l}^0(\pi_0(W(g_l))) \prod_{k=1}^{m} \alpha_{is_k}^0(\pi_0(W(f_k)))\,\Omega_0 \end{aligned} \tag{2.48}$$

we obtain

$$j_A\left(\prod_{k=1}^{m} {}^A\alpha_{it_k}^0({}^A\pi_0(W(g_k)))\right) j_A^* = \prod_{k=1}^{m} \alpha_{it_k}^0(\pi_0(W(g_k))) \tag{2.49}$$

Therefore applying Lemma 2.6 again, we conclude that

$$j_A({}^A\mathbf{m}_0)\, j_A^* = \pi_0(\mathscr{W}(\mathbf{h}))'' \tag{2.50}$$

Let us observe also that the map

$$\begin{aligned}{}^A\tilde{\pi}_0 \colon W\left(\sum_\alpha e^{it_\alpha h\mu} f_\alpha\right) \\ \to \prod_{1 \leqslant \alpha < \beta \leqslant n} \exp\{i\sigma(e^{it_\alpha h} f_\alpha, e^{it_\beta h} f_\beta) \\ \times \prod_\alpha {}^A\alpha^0_{t_\alpha}({}^A\pi_0(W(f_\alpha)))\end{aligned} \tag{2.51}$$

can be extended to representation of the full Weyl algebra $\mathscr{W}(\mathbf{h})$ in $L({}^A\mathscr{H}_0)$ and moreover the obtained representation extends ${}^A\pi_0'$. For this, let us observe that

$$\begin{aligned}&{}^A\tilde{\pi}_0\left(W\left(\sum_\alpha e^{it_\alpha h^\mu} f_\alpha\right)\right) \\ &= \prod_{1 \leqslant \alpha < \beta \leqslant n} \exp\{i\sigma(e^{it_\alpha h^\mu} f_\alpha, e^{it_\beta h^\mu} f_\beta)\} \prod_\gamma {}^A\alpha^0_{t_\gamma}({}^A\pi_0(W(f_\gamma))) \\ &= \prod_{1 \leqslant \alpha < \beta \leqslant n} \exp\{i\sigma(e^{it_\alpha h^\mu} f_\alpha, e^{it_\beta h^\mu} f_\beta)\}\, j_A^{-1}\left(\prod_\gamma \alpha^0_{t_\gamma}(\pi_0(W(f_\gamma)))\right) j_A \\ &= j_A^{-1}\left(\pi_0\left(W\left(\sum_\gamma e^{it_\gamma h^\mu} f_\gamma\right)\right)\right) j_A\end{aligned} \tag{2.52}$$

by using (2.49), and the fact that π_0 is a representation of $\mathscr{W}(\mathbf{h})$. From Lemma 2.6 we know that for any $g \in L_2^r(\mathbb{R}^d)$ there exists a sequence

$$(t_1^\alpha, \ldots, t_{n_\alpha}^\alpha), (f_1^\alpha, \ldots, f_{n_\alpha}^\alpha) \in L_2(\mathbb{R}^d)$$

such that $\sum_k \exp\{it_k^\alpha h^\mu\} f_k^\alpha \to g$ in $L^2(\mathbb{R}^d)$ sense.

Because π_0 is an $L_2(\mathbb{R}^d)$ continuous representation of $\mathscr{W}(\mathbf{h})$, it follows that the limit

$$\lim_{\alpha \to \infty} j_A^{-1}\left(\pi_0\left(W\left(\sum_k e^{it_k^\alpha h^\mu} f_k^\alpha\right)\right)\right) j_A$$

exists in the weak sense, and therefore we conclude that also

$$\begin{aligned}\lim_{\alpha \to \infty} \prod_{1 \leqslant k_\alpha < l_\alpha \leqslant n} \exp\{i\sigma(\exp(it_{k_\alpha}^\alpha h^\mu) f_{k_\alpha}^\alpha, \exp(it_{l_\alpha}^\alpha h^\mu) f_{l_\alpha}^\alpha)\} \\ \times \prod_\gamma {}^A\alpha^0_{it_\gamma}({}^A\pi_0(W(f_\gamma))) \equiv {}^A\tilde{\pi}(W(g))\end{aligned} \tag{2.53}$$

exists in the weak sense. Now it is easy to check that ${}^{A}\tilde{\pi}$ as defined in (2.53) is really a *-bounded representation of $\mathscr{W}(\mathbf{h})$ in $L({}^{A}\mathscr{H}_0)$ and such that ${}^{A}\tilde{\pi}_{0|\mathscr{A}(\mathbf{h})} = {}^{A}\pi_0$. ∎

Step 4. Proof of Lemma 2.6. The operator $e^{it\Delta}$ acts as $e^{it\Delta}f = (e^{itp^2}\hat{f})^{\vee}$, where $^\wedge$ and $^\vee$ denote the Fourier transform and its inverse. Let us take $g \in C_c(\mathbb{R}^d)$; which is a dense subspace in $L^2(\mathbb{R}^d)$. Let

$$g_1(p) = \frac{1}{2}\left[g(p) + \overline{g(-p)}\right] \qquad \text{and} \qquad g_2(p) = \frac{1}{2i}\left[g(p) - \overline{g(-p)}\right]$$

be Hermitian parts of g. Because g_1 is the Fourier transform of a real-valued function, we may write

$$\begin{aligned}\left\| g(p) - \left[\sum_{k=1}^{n} \hat{f}_k(p)\, e^{it_k p^2} - \sum_{k=1}^{n} \hat{f}_k(p)\, e^{it_k p^2} - g_1(p)\right]\right\|_{L^2} \\ = \left\| ig_2(p) - i\sum_{k=1}^{n} \hat{f}_k(p)\sin(t_k p^2)\right\|_{L^2} \end{aligned} \tag{2.54}$$

so it is enough to show that for every $\varepsilon > 0$ there exist real-valued functions $f_1,..., f_n \in L^2(\mathbb{R}^d)$ and $t_1,..., t_n \in \mathbb{R}$ such that

$$\left\| g_2(p) - \sum_{k=1}^{n} \hat{f}_k(p)\sin(t_k p^2)\right\|_{L^2} < \varepsilon \tag{2.55}$$

Let B denote a ball in $\mathbb{R}^d$ of radius $c > 0$ such that $\operatorname{supp} g(p) \subset B$. Let $\hat{f}_k(p) = a_k(p)\, g_2(p)$, where $a_k(p) \in C_0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $a_k(p) = \overline{a_k(p)} = a_k(|p|)$. It is clear that $\hat{f}_k(p)$ is Hermitian and belongs to $L^2(\mathbb{R}^d)$. Then

$$\begin{aligned}\left\| g_2(p) - \sum_{k=1}^{n} \hat{f}_k(p)\sin(t_k p^2)\right\|_{L^2} \\ \leqslant \|g_2\|_\infty \left\| 1 - \sum_{k=1}^{n} a_k(p)\sin(t_k p^2)\right\|_{L^2(B)}\end{aligned} \tag{2.56}$$

Let us deform the constant function 1 to a function $f_0 \in C(B)$ such that $f_0(p) \geqslant 0\ \forall_{p \in B}$, $f_0(0) = 0$, and $\|1 - f_0\|_{L^2(B)} < \varepsilon$, $f_0(p) = f_0(p')$ if $|p| = |p'|$. Then

$$\begin{aligned}\left\| 1 - \sum_{k=1}^{n} a_k(p)\sin(t_k p^2)\right\|_{L^2(B)} \\ \leqslant \varepsilon + \mu(B)^{1/2} \sup_{|p| \in [0,\, c]} \left|f_0(|p|) - \sum_{k=1}^{n} a_k(|p|)\sin(t_k p^2)\right|\end{aligned} \tag{2.57}$$

where $\mu(B)$ is the Lebesque measure of the ball B.

We consider a real algebra generated by $\sum_{k=1}^{n} a_k(|p|)\sin(t_k\,|p|^2)$ on $(0, c]$. It is clear that $\sin(t\,|p|^2)$ separates points in $(0, c]$ and for every $|p| \in (0, c]$ there exists $t \in \mathbb{R}$ such that $\sin t\,|p|^2 \neq 0$. Because we may choose $a(p)$ such that $\alpha_{|B} = 1$, our algebra separates points and nowhere vanishes in $(0, c]$. Thus, applying the Stone–Weierstrass theorem to $C_0(0, c]$, we have that

$$\sup_{|p| \in (0, c]} \left|f_0(|p|) - \sum_{k=1}^{n} a_k(|p|)\sin(t_k\,|p|^2)\right| < \varepsilon$$

for some $a_1, ..., a_n$ and $t_1, ..., t_n$. Finally,

$$\left\| g_2(p) - \sum_{k=1}^{n} \hat{f}_k(p)\sin(t_k\, p^2) \right\|_{L^2} \leqslant \|g_2\|_\infty\,(1 + \mu(B)^{1/2})\,\varepsilon$$

which proves the assertion for $\mathscr{E}(p) = p^2$. The same proof works for $\mathscr{E}(p) = (p^2 + m^2)^{1/2}$, $m \geqslant 0$. ∎

To exploit the stochastic positivity EG(6) of the system ${}^{AE}\mathbb{G}$ and for the further development we shall introduce two basic concepts of the generalized thermal process and the generalized thermal random field.

It should be emphasized that these concepts are heavily inspired by the abstract theory developed by Klein and Landau[25] (see also ref. 32).

Definition 2.7. Any generalized, periodic (with the period β) stochastic process $(\xi_t)_{t \in \mathbb{R}}$ with values in $\mathscr{D}'(\mathbb{R}^d)$ will be called a thermal process (with the temperature β) iff:

Tp(1) The process $(\xi_t)_{t \in \mathbb{R}}$ is symmetric on K_β, i.e.,

$$\forall_{-\beta/2 \leqslant \tau \leqslant \beta/2}\ \forall_{f \in \mathscr{D}(\mathbb{R}^d)} \langle \xi_\tau, f\rangle = \langle \xi_{-\tau}, f\rangle \qquad \text{(in law)} \tag{2.58}$$

Tp(2) The process $(\xi_t)_{t \in \mathbb{R}}$ is (locally) homogeneous, i.e.,

$$\forall_{\substack{\tau, s \in K_\beta \\ \tau + s \leqslant \beta/2}}\ \forall_{f \in \mathscr{D}(\mathbb{R}^d)} \langle \xi_{\tau+s}, f\rangle = \langle \xi_\tau, f\rangle \qquad \text{(in law)} \tag{2.59}$$

Tp(3) The process $(\xi_t)_{t \in \mathbb{R}}$ is OS-positive on K_β, i.e., for any bounded $F \in C_b(\mathbb{R}^n)$, any $\underset{\sim}{\tau}^n \in [0, \beta/2]^{\times n}$, $\underset{\sim}{f}^n \in \mathscr{D}(\mathbb{R}^d)^{\times n}$,

$$0 \leqslant EF(\langle \xi_{-\tau_1^n}, f_1\rangle, ..., \langle \xi_{-\tau_n^n}, f_n\rangle)\, F(\langle \xi_{\tau_1^n}, f_n\rangle, ..., \langle \xi_{\tau_n^n}, f_n\rangle) \tag{2.60}$$

Tp(4) The moments

$$E\left(\prod_{i=1}^{n} \exp(i\langle \xi_{\tau_i}, f_i\rangle)\right) \equiv G^{(\xi)}_{f_1,...,f_n}(\tau_1, ..., \tau_n) \tag{2.61}$$

are continuous in $\underset{\sim}{\tau}^n \in (K_\beta)^{\times n}$ and on $\mathscr{D}(\mathbb{R}^d)^{\times n}$.

A thermal process ξ is called Euclidean invariant if additionally:

Tp(5) The moments (2.61) are invariant under the action of the Euclidean group $E(d)$ in $\mathscr{D}(\mathbb{R}^d)$.

A thermal process $(\xi_t)_{t \in \mathbb{R}}$ is called tempered iff the moments (2.61) are continuous on $S(\mathbb{R}^d)^{\times n}$, L_p-continuous iff the moments (2.61) are continuous on $L^p(\mathbb{R}^d)^{\times n}$, etc.

If $(\xi_t)_t$ is a generalized thermal process, then its corresponding path space measure construction leads to the concept of the random generalized field.

Definition 2.8. Any generalized random field μ^β on $\mathscr{D}'(K_\beta \times \mathbb{R}^d)$ [i.e., any probabilistic, Borel cylindric (PBC) measure] will be called a generalized thermal random field iff:

Tf(1) We have

$$\forall_{\substack{g \in C^\infty(K_\beta) \\ f \in \mathscr{D}(\mathbb{R}^d)}} \langle \phi; r(g \otimes f) \rangle = \langle \phi; g \otimes f \rangle \qquad \text{(in law } \mu^\beta) \tag{2.62}$$

where $r(g \otimes f)(\tau, x) = g(-\tau) f(x)$.

Tf(2) We have

$$\forall_{\substack{g \in C_0^\infty(K_\beta) \\ f \in \mathscr{D}(\mathbb{R}^d)}} \langle \phi; t_s(g \otimes f) \rangle = \langle \phi; g \otimes f \rangle \qquad \text{(in law } \mu^\beta) \tag{2.63}$$

for any $s > 0$ such that supp $t_s(g) \subseteq [-\beta/2, \beta/2]$, where $t_s(g)(\tau) \equiv g(t+s)$.

Tf(3) The field μ^β is OS-positive on the circle K_β, i.e., for any bounded cylindric function F based on $(g_1 \otimes f_1, ..., g_n \otimes f_n)$, where $g_i \in C^\infty[0, \beta/2]$ for all i, $f_i \in \mathscr{D}(\mathbb{R}^d)$, we have

$$\begin{aligned} 0 \leqslant \mu^\beta(RF(\langle \phi, g_1 \otimes f_1 \rangle, ..., \langle \phi, g_n \otimes f_n \rangle) \\ \times F(\langle \phi, g_1 \otimes f_1 \rangle, ..., \langle \phi, g_n \otimes f_n \rangle)) \end{aligned} \tag{2.64}$$

where

$$\begin{aligned} RF(\langle \phi, g_1 \otimes f_1 \rangle, ..., \langle \phi, g_n \otimes f_n \rangle) \\ = F(\langle \phi, rg_1 \otimes f_1 \rangle, ..., \langle \phi, rg_n \otimes f_n \rangle) \end{aligned} \tag{2.65}$$

Tf(4) For any $\tau \in K_\beta$ the random elements $\langle \phi, \delta_\tau \otimes f \rangle$ [defined as unique limits in $L^p(d\mu^\beta)$ sense $\lim_{\varepsilon \downarrow 0} \langle \phi, \delta_\tau^\varepsilon \otimes f \rangle$, for any mollifier $\delta_\tau^\varepsilon \to \delta_\tau$] exist and moreover the moments

$$\mu^\beta \left(\prod_{i=1}^n \exp(i \langle \phi, \delta_{\tau_i} \otimes f_i \rangle) \right) \equiv G^{(\mu)}_{f_1, ..., f_n}(\tau_1, ..., \tau_n) \tag{2.66}$$

are continuous in $\underset{\sim}{\tau}^n \in K_\beta^{\times n}$; $\underset{\sim}{f}^n \in \mathscr{D}(\mathbb{R}^d)^{\times n}$.

Tf(5) A generalized random thermal field μ is Euclidean invariant iff the moments $G^{(\mu)}$ are invariant under the natural action of $E(d)$ in $\mathscr{D}(\mathbb{R}^d)$.

Additionally. A generalized random thermal field μ will be called tempered iff the moments (2.66) are tempered distributions, L^p-continuous iff the moments (2.66) are L^p-continuous, etc.

Proposition 2.9. 1. Let $(\xi_t)_t$ be a tempered thermal process with the temperature β. There exists a unique (up to the unitary equivalence) W^*-KMS structure

$$\mathbb{C}^\xi = (\mathscr{H}^\xi, \Omega^\xi; \alpha_t^\xi; \pi^\xi: \mathscr{A}(S(\mathbb{R}^d)) \to L(\mathscr{H}^\xi), \mathbf{m}^\xi)$$

where

$$\mathbf{m}^\xi = W^* - \{\alpha_{t_1}^\xi(\pi^\xi(W(f_1))) \cdots \alpha_{t_n}^\xi(\pi^\xi(W(f_n)))\}$$

with $f_i = \bar{f}_i \in S(\mathbb{R}^d)$ real, whose Euclidean Green functions restricted to $\pi^\xi(\mathscr{A}(S(\mathbb{R}^d)))$ coincide with the moments $G^\xi_{\cdot,\dots,\cdot}(\tau_1,\dots,\tau_n)$, i.e., for any $-\beta/2 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant \beta/2$,

$$\begin{aligned} &\langle \Omega^\xi; \alpha_{i\tau_n}^\xi(\pi^\xi(W(f_n))) \cdots \alpha_{i\tau_1}^\xi(\pi^\xi(W(f_1)))\, \Omega^\xi \rangle \\ &\quad = G^\xi_{f_1,\dots,f_n}(\tau_1,\dots,\tau_n) \\ &\quad = E e^{i\langle \xi_1, f_1\rangle} \cdots e^{i\langle \xi_n, f_n\rangle} \end{aligned} \tag{2.67}$$

2. Let μ be a tempered thermal field (at the temperature β). There exists a unique (up to a unitary equivalence) W^*-KMS structure

$$\mathbb{C}^{(\mu)} = (\mathscr{H}^{(\mu)}, \Omega^{(\mu)}; \alpha_t^{(\mu)}; \pi^{(\mu)} \in \mathrm{Hom} * (A(S(\mathbb{R}^d)), L(\mathscr{H}^{(\mu)})); \mathbf{m}^{(\mu)})$$

where

$$\mathbf{m}^{(\mu)} = W^* - \{\alpha_{t_1}^{(\mu)}(\pi^{(\mu)}(W(f_1))) \cdots \alpha_{t_n}^{(\mu)}(\pi^{(\mu)}(W(f_n)))\}$$
$$t_1,\dots,t_n \in \mathbb{R}, \qquad f_1,\dots,f_n \in S(\mathbb{R}^d); \qquad f_i = \bar{f}_i$$

whose Euclidean Green function restricted to $\pi^{(\mu)}(\mathscr{A}(S(\mathbb{R}^d)))$ coincides with $G^\mu_{\cdot,\dots,\cdot}(\tau_1,\dots,\tau_n)$.

3. If the tempered random thermal field μ is the path space measure of a tempered process $(\xi_t)_t$, i.e., if

$$\begin{aligned} &E \exp(i\langle \xi_{\tau_1}, f_1\rangle) \cdots \exp(i\langle \xi_{\tau_n}, f_n\rangle) \\ &\quad = \mu(\exp(i\langle \phi, \delta_{\tau_1} \otimes f_1\rangle) \cdots \exp(i\langle \phi, \delta_{\tau_n} \otimes f_n\rangle)) \end{aligned} \tag{2.68}$$

for all $\tau_1,\dots,\tau_n \in K_\beta$; $f_1,\dots,f_n \in S(\mathbb{R}^d)$, then the W^*-KMS systems $\mathbb{C}^{(\xi)}$ and $\mathbb{C}^{(\mu)}$ coincide.

Proof. Let $(\xi_t)_t$ be a given tempered thermal process at the temperature $\beta > 0$. It follows from Definition 2.7 that the moments $G^{(\xi)}_{f_1,\ldots,f_n}(\tau_1,\ldots,\tau_n)$ define on the Abelian sector $\mathscr{A}(S(\mathbb{R}^d))$ of the Weyl algebra $\mathscr{W}(S(\mathbb{R}^d))$ a system of functions fulfilling EG(1)–EG(6) with possible lack of EG(5)(ii) and with modified EG(1)(ii):

EG(1)(ii)′. The functionals

$$G^{(\xi)}_{\cdot,\ldots,\cdot}(\tau_1,\ldots,\tau_n): S(\mathbb{R}^d)^{\times n} \ni (f_1,\ldots,f_n) \mapsto G^{(\xi)}_{f_1,\ldots,f_n}(\tau_1,\ldots,\tau_n)$$

are continuous and $|G^{(\xi)}_{f_1,\ldots,f_n}(\tau_1,\ldots,\tau_n)| \leqslant 1$.

Also, EG(3) should be properly modified. All these modifications, however, do not affect seriously the construction presented in step 1 of Proposition 2.5. Proceeding analogously to step 1 of Proposition 2.5, we can construct $\mathbb{C}^{\xi}$. Similarly we prove the existence of $\mathbb{C}^{\mu}$. The identification of $\mathbb{C}^{\xi}$ and $\mathbb{C}^{\mu}$ follows from 2.69 and the uniqueness part of 1 and 2. ∎

It follows from the results of ref. 32, stochastic positivity EG(6), and Proposition 2.5 that the thermal structure of the free Bose gas can be described fully in terms of the corresponding stochastic thermal structures.

Proposition 2.10. Let $\mathscr{E}(p)$ be given by (2.1) and let $0 < z$ be such that $\sup_p z\exp\{-\beta\mathscr{E}(p)\} < 1$. Then for any $\beta > 0$:

1. There exists a unique (up to a stochastic equivalence) Gaussian thermal process $(\xi^0_t)_{t\in\mathbb{R}}$ with values in $\mathscr{D}'(\mathbb{R}^d)$ such that

$$E\langle \xi^0_t, f\rangle = 0; \qquad E(\langle \xi^0_t, f\rangle\langle \xi^0_{t'}, g\rangle) = S^{\beta}_0(|t-t'|, f\otimes g) \tag{2.69}$$

The process $(\xi^0_t)_{t\in\mathbb{R}}$ is Euclidean invariant, ergodic, and L^2-continuous.

2. There exists a unique (up to a stochastic equivalence) Gaussian generalized thermal random field μ^{β}_0 such that

$$\begin{gathered}\mu^{\beta}_0(\langle\phi, f\rangle) = 0\\ \mu^{\beta}_0(\langle\phi, \delta_\tau\otimes f\rangle\langle\phi, \delta_{\tau'} f\rangle) = S^{\beta}_0(|\tau-\tau'|, f\otimes g)\end{gathered} \tag{2.70}$$

The thermal field μ^{β}_0 is Euclidean invariant, ergodic, and L^2-continuous.

3. The generalized random field μ^{β}_0 can be identified with the path space measure of the process $(\xi^0_t)_{t\in\mathbb{R}}$, i.e., for any bounded, cylindric function F with base $(\tau_1, f_1),\ldots,(\tau_n, f_n)$

$$\begin{aligned}&EF(\langle\xi_{\tau_1}, f_1\rangle,\ldots,\langle\xi_{\tau_n}, f_n\rangle)\\ &\quad\equiv \mu^{\beta}_0(F(\langle\phi, \delta_{\tau_1}\otimes f_1\rangle,\ldots,\langle\phi,\delta_{\tau_n}\otimes f_n\rangle))\end{aligned} \tag{2.71}$$

4. Let ν_0^β be a Gaussian measure on $\mathscr{D}'(\mathbb{R}^d)$ with mean zero and the covariance given by

$$\nu_0^\beta(\langle \varphi, f \rangle \langle \varphi, g \rangle) = C_0^\beta(f \otimes g) \tag{2.72}$$

Then the measure ν_0^β is the *unique* stationary measure of the process $(\xi_t^0)_{t \in \mathbb{R}}$ and ν_0^β is equal to the restriction of μ_0^β to the σ-algebra at $\tau = 0$, i.e., $\mu^\beta_{0\,|\,\Sigma(0)} = \nu_0^\beta$, where

$$\Sigma(0) = \sigma\{ \langle \phi, \delta_0 \otimes f \rangle ; f \in f \in \mathscr{D}(\mathbb{R}^d)\}$$

Moreover, the measure ν_0^β is quasiinvariant under the translations by $\mathscr{D}(\mathbb{R}^d)$.

Remarks. Other well-known examples of generalized thermal processes arise in the study of two-dimensional models of Euclidean (quantum) field theory[23, 33] and also in the context of the Euclidean version of the Bisognano–Wichman theorem.[33, 34] Similar stochastic thermal structures on the Abelian sectors of the corresponding algebras of observables also appear in the context of (an)harmonic lattice crystals[21, 26, 27] and certain spin systems.[24, 35]

The common problem of all these examples is to construct a modular structure on whole algebra of observables from arising stochastic thermal structures on the Abelian sector. In the case of the free Bose gas the complete solution of this problem is given by Proposition 2.5.

From the assumption $\sup_p |z \exp\{-\beta \mathscr{E}(p)\}| < 1$ it follows that the operator $(1 - z\exp\{-\beta\mathscr{E}(p)\})^{-1}$ exists in $L^2(\mathbb{R}^d)$ and is bounded, strictly positive, and self-adjoint. Let $\mathbf{h}^\beta(\mathbb{R}^d)$ be the metric completion of the space $\mathscr{D}(\mathbb{R}^d)$ equipped with the inner product

$$\langle f, g \rangle \equiv \int \overline{f(x)} (1 - z e^{-\beta\mathscr{E}(p)})^{-1}\,(x - y)\, g(y)\, dx\, dy \tag{2.73}$$

From the simple estimates

$$\|f\|_{L^2(\mathbb{R}^d)} \leqslant \|f\|_\beta \leqslant (\inf_p (1 - z e^{-\beta\mathscr{E}(p)}))^{-1}\, \|f\|_{L^2(\mathbb{R}^d)} \tag{2.74}$$

it follows that $\mathbf{h}^\beta_{\bullet}$ is essentially equal to $L_2(\mathbb{R}^d)$. Using the $L^2(\mathbb{R}^d)$-continuity of the process ξ_t^0 and estimates (2.74), we can define a version $\tilde{\xi}_t^0$ of ξ_t^0 obtained by extension of the index space $\mathscr{D}(\mathbb{R}^d)$ onto the space $\mathbf{h}^\beta$. The new process $\tilde{\xi}_t^0$ is indexed by $K_\beta \times \mathbf{h}^\beta(\mathbb{R}^d)$. For any Borel subset $I \subset K_\beta$ we denote by $\Sigma(I)$ the smallest σ-algebra generated by $\{(\tilde{\xi}_t^0, f) \,|\, t \in J, f \in \mathbf{h}^\beta(\mathbb{R}^d)\}$. For any $t, s \in K_\beta$ we will denote by $[t, s]$ the closed interval from t to s in the counterclockwise direction. The corresponding conditional expectations with respect to the σ-algebras $\Sigma[t, s]$ $[\Sigma(J)]$ will be denoted by $E^0_{[t,s]}$ (resp. E^0_J).

Proposition 2.11. 1. For any allowed form of $\mathscr{E}(p)$, z such that $|ze^{-\mathscr{E}(p)}| < 1$ the corresponding free thermal process $\tilde{\xi}^0_t$ has two-sided Markov property on K_β in the sense that

$$E^0_{[s,r]} \circ E^0_{[r,s]} = E^0_{\{r,s\}} \circ E^0_{[r,s]} \tag{2.75}$$

2. Let $\Xi(J) \equiv \sigma\{\phi(t, f) \mid t \in J, f \in \mathbf{h}^\beta\}$ be the corresponding σ-algebras in $B(\mathscr{D}'(K_\beta \times \mathbb{R}^d))$ and let $\tilde{E}^0(J)$ denote the corresponding conditional expectation values. Then the free thermal random field μ^β_0 has the following two-sided Markov property on K_β:

$$\tilde{E}^0_{[s,r]} \circ \tilde{E}^0_{[r,s]} = \tilde{E}^0_{\{r,s\}} \circ \tilde{E}^0_{[r,s]} \tag{2.76}$$

Proof. It follows easily that the operator $h^\mu = h_0 + \mu 1$ is a non-negative self-adjoint operator in $\mathbf{h}^\beta$ [on the same domain as in $L^2(\mathbb{R}^d)$]. Moreover, the covariance operator $\Gamma^\beta_0(t)$ of the process $\tilde{\xi}^0_t$ indexed by $K_\beta \times \mathbf{h}^\beta_r$ is given by

$$\Gamma^\beta_0(t) = e^{-th^\mu} + e^{-(\beta - t)h^\mu} \tag{2.77}$$

Applying Theorem 4.1 of ref. 32, we conclude the proof of the first part. The second part follows easily by identification of $\phi(t, x)$ with $\xi_t(x)$ given in Proposition 2.10 and the density of $\mathscr{D}(\mathbb{R}^d)$ in the space $\mathbf{h}^\beta_r(\mathbb{R}^d)$. ∎

2.2. Local aspects [the case $\mathscr{E}(p) = p^2$]

Let $\Lambda \subset \mathbb{R}^d$ be a bounded region with a boundary $\partial\Lambda$ of a class at least C^1-piecewise. Then, for any $b \in C(\partial\Lambda)$ the self-adjoint extension $-\Delta^b_\Lambda$ of the symmetric operator $-\Delta$ defined on $C^\infty_0(\Lambda)$ can be constructed. It is well known that the arising semigroup $\{\exp(-t\Delta^b_\Lambda, t \geqslant 0\}$ is positivity-preserving on $L_2(\Lambda)$; therefore there exists a stationary Markov process $\mathscr{B}^\sigma_\Lambda(t)$ with independent increments, with values in $\bar{\Lambda}$ for which the kernel $K^{(\Lambda,b)}_t$ of $\exp(-t\Delta^b_\Lambda)$ plays the role of the transition function.

Let $\mathscr{W}_\Lambda$ be the local Weyl algebra built over the space $L_2(\Lambda)$ and let $\mathscr{W}^F_\Lambda$ be its Fock-space realization in the Fock–Bose space $\Gamma_{-1}(L_2(\Lambda))$. In particular, we have

$$W^F_\Lambda(f) = \exp\{i[a^+_\Lambda(\bar{f}) + a_\Lambda(f)]\} = \exp[i\varphi_\Lambda(f)] \tag{2.78}$$

where a_Λ and a^+_Λ are standard annihilation and creation operators in $\Gamma_{-1}(L_2(\Lambda))$.

Let $P^{(\Lambda, b)}(d\lambda)$ be the spectral measure for the operator $-\Delta_\Lambda^b$. Then, we can define the finite-volume thermal state $\omega_0^{(\Lambda, b)}$ on $\mathscr{W}_\Lambda^F$ by the formula

$$\omega_0^{(\Lambda, b)}(W_F(f)) = \exp - \tfrac{1}{2} C^\beta_{0, (\Lambda, b)}(f) \tag{2.79}$$

where

$$C^\beta_{0, (\Lambda, b)}(f) \equiv \langle f \mid C^\beta_{0, (\Lambda, b)}(f) \rangle_{L_2(\Lambda)} \tag{2.80}$$

$$\hat{C}^\beta_{0, (\Lambda, b)}(f) = \int P^{(\Lambda, b)}(d\lambda) \frac{1 + ze^{-\beta\lambda}}{1 - ze^{-\beta\lambda}} \tag{2.81}$$

It is well known (see, e.g., ref. 17) that for any monotonic sequence $(\Lambda_n)_n$ of bounded regions in $\mathbb{R}^d$ and with sufficiently regular boundaries $\partial\Lambda_n$ tending to $\mathbb{R}^d$ by inclusion and for any sequence $b_{\partial\Lambda_n} \in C(\partial\Lambda_n)$ we have the weak convergence

$$\lim_{n \to \infty} \hat{C}^\beta_{0, (\Lambda_n, b_n)} = C_0^\beta \qquad \text{if} \quad z \in (0, 1)$$

The corresponding GNS construction applied to $(\mathscr{W}_\Lambda^F(\cdot\,; \omega_0^{(\Lambda, b)})$ leads again to the W^*-KMS system

$$\mathbb{C}_0^{(\Lambda, b)} = (\mathscr{H}_0^{(\Lambda, b)}; \pi_0^{(\Lambda, b)}; \Omega_0^{(\Lambda, b)}; \alpha_0^{(\Lambda, b)}; \pi_0^{(\Lambda, b)}(\mathscr{W}_\Lambda^F)'')$$

and the corresponding Green functions can again be easily computed and the analycity properties similar to those of $\mathbb{G}_0$ established. In particular, the corresponding Euclidean Green functions ${}^E\mathbb{G}_0(\Lambda, b_{\partial\Lambda})$ again fulfill the system of axioms EG(1)–EG(4) and EG(6); therefore the whole discussion from Section 2.1 can be repeated with obvious modifications.

Lemma 2.12. Let $z = e^{-\beta\mu}$ be sufficiently small and let (Λ_n) be a monotonic sequence of bounded convex regions in $\mathbb{R}^d$ with boundaries $\partial\Lambda_n$ of class at least C^3 and with mean curvatures uniformly bounded. Then for any choice of $b_n \in C(\partial\Lambda_n)$, any $f_1,..., f_m \in L_2(\Lambda)$, $\underset{\sim}{s}^m \in T_n^\beta$ we have the convergence

$$\begin{aligned} &\lim_{n \to \infty} {}^E G_0(\Lambda_n, b_n)((s_1, f_1),..., (s_m, f_m)) \\ &\qquad = {}^E G_0^0((s_1, f_1),..., (s_m, f_m)) \end{aligned} \tag{2.82}$$

Proof. The monotonicity in the boundary conditions:
If $b_1(x) \leqslant b_2(x)$ for all $x \in \partial\Lambda$, then

$$K_t^{(\Lambda, b_1)}(x, y) \geqslant K_t^{(\Lambda, b_2)}(x, y) \tag{2.83}$$

for all $x, y \in \Lambda$, $t > 0$. Therefore

$$\sup_{b \in C(\partial\Lambda)} |\mathscr{K}_t^{(\Lambda, b)}(x, y) - \mathscr{K}_t(x, y)| = |\mathscr{K}_t^{(\Lambda, 0)}(x, y) - \mathscr{K}_t(x, y)| \quad (2.84)$$

for all $t, x, y \in \Lambda$, where $\mathscr{K}_t^{(\Lambda, 0)}$ is the kernel of the semigroup $\{\exp(-t\,\Delta_\Lambda^N)(x, y),\, t \geqslant 0\}$, where Δ_Λ^N corresponds to the Neumann boundary condition. By the (rough) estimate of ref. 36, we have with our assumptions on (Λ_n)

$$\begin{aligned} &|\mathscr{K}_t^{(\Lambda_n, b^{\partial\Lambda_n})}(x, y) - \mathscr{K}_t(x, y)| \\ &\quad \leqslant C e^{\lambda t} t^{-d/2} \exp\left\{-c\left(\frac{d(x, \Lambda_n^c)^2 + d(y, \Lambda_n^c)^2}{4t}\right)\right\} \end{aligned} \quad (2.85)$$

for all $x, y \in \Lambda$, $t \in \mathbb{R}$, where C, c, and $\lambda \geqslant 0$ are some constants.

It is due to the quasifree nature of the states $\omega_0^{(\Lambda, b)}$ that it is enough to consider the one-time Green function only,

$$\begin{aligned} &|{}^E G^0(\Lambda_n, b_n)((0, f_1), (s_1, f_2)) - {}^E G^0((0, f_1), (s_1, f_2))| \\ &\quad \leqslant |\exp(i\sigma(f_1, e^{is_1 h^\mu(\Lambda_n, b_n)} f_2))| \\ &\quad\quad \times |S_0^\beta(s_1 \,|\, f_2 \otimes f_2) - \exp\{i[\sigma(f_1, e^{is_1 h_\Lambda^\mu} f_2) - \sigma(f_1, e^{is_1 h^\mu} f_2)]\} \\ &\quad\quad \times S_0^\beta(s_1, f_1 \otimes f_2)| \end{aligned} \quad (2.86)$$

It is well known that

$$\lim_{n \to \infty} \exp[it(-\Delta_{\Lambda_n}^{b_{\Lambda_n}} + \mu 1)] = \exp[it(-\Delta + \mu 1)]$$

strongly in $L^2(\mathbb{R}^d)$; therefore we shall omit the symplectic factor in the last formula, concentrating attention on

$$\begin{aligned} &|{}^{(\Lambda_n, b_n)} S_0^\beta(s \,|\, f_1 \otimes f_2) - S_0^\beta(s \,|\, f_1 \otimes f_2)| \\ &\quad \leqslant \sum_{n \geqslant 0} z^{n + s/\beta} \int dx\, dy\, f_1(x)\, f_2(y)\, |\mathscr{K}_{(\beta n + s)}^{(\Lambda_n, b_n)}(x, y) - \mathscr{K}_{(\beta n + s)}(x, y)| \\ &\quad\quad + \sum_{n \geqslant 0} z^{n + 1 - s/\beta} \int dx\, dy\, f_1(x)\, f_2(y) \\ &\quad\quad \times |\mathscr{K}_{(\beta(n+1) - s)}^{(\Lambda_n, b_n)}(x, y) - \mathscr{K}_{(\beta(n+1) - s)}(x, y)| \end{aligned} \quad (2.87)$$

Therefore localizing first f_1, f_2 and taking into account (2.85), we obtain

$$\lim_{n \to \infty} {}^{(\Lambda_n, b_n)} S_0^\beta(s, f_1 \otimes f_2) = S_0^\beta(s \,|\, f_1 \otimes f_2) \quad (2.88)$$

provided $e^{-\beta\mu} e^{\lambda} < 1$. ∎

Remarks. The restriction $e^{-\beta\mu}e^{\lambda}<1$ is no doubt only an artifact of the rough estimate (2.85) used. It is natural to expect that actually this lemma is valid for all $0<z<1$. For a Dirichlet boundary condition the constant λ can be taken to be equal to zero and this gives the result of the independence of the limiting Green functions of the Dirichlet boundary condition in the full noncritical interval $z\in(0,1)$.

The finite-volume, conditional thermal processes (resp. thermal random fields) will be denoted by $\xi_t^{(\Lambda, b_{\partial\Lambda})}$ (resp. $\mu_0^{(\Lambda, b_{\partial\Lambda})}$).

Having established properties EG(1)–EG(4) of the corresponding Euclidean Green functions ${}^{E}\mathbb{G}_0^{(\Lambda, b_{\partial\Lambda})}$ (resp. ${}^{AE}\mathbb{G}_0^{(\Lambda, b_{\partial\Lambda})}$), we can construct again three different *a priori* W^*-KMS structures: ${}^{E}\mathbb{C}_0^{(\Lambda, b_{\partial\Lambda})}$, ${}^{EA}\mathbb{C}_0^{(\Lambda, b_{\partial\Lambda})}$, and the basic GNS system $\mathbb{C}_0^{(\Lambda, b_{\partial\Lambda})}$. It appears that all the claims of a properly modified Proposition 2.5 are still valid and the proof is almost identical, with the exception of Lemma 2.6, which is replaced by Lemma 2.13.

Let Λ be a bounded, open, and connected region in $\mathbb{R}^d$, $d\geqslant 2$, with a smooth boundary. Let us define $-\Delta_\Lambda^b(f)=-\Delta f$ for $f\in C_0^2(\Lambda)$, where $\mathscr{D}(-\Delta_\Lambda^b)$ consists of those $f\in L^2(\Lambda)$ which satisfy the following:

(a) $f\in C^2(\Lambda)$

(b) $\partial^n f(x)=b(x)\,f(x)$ for $x\in\partial\Lambda$

with ∂^n being the normal inward derivative. It follows that $-\Delta_\Lambda^b$ for $b\in C^1(\partial\Lambda)$ is densely defined, symmetric, and strongly positive. Let $\hat{L}^b$ be the Friedrichs extension of $-\Delta_\Lambda^b$ to a self-adjoint operator. Then, as is well known (see, e.g., ref. 37) the spectrum of a self-adjoint L^b is purely discrete and all eigenfunctions of L^b are real-valued. Moreover, the semigroup $\exp(-tL_\Lambda^b)$ is of trace class.

It is well known (see, e.g., ref. 37) that $-\hat{L}_\Lambda^b$ possesses real-valued eigenfunctions $\{u_k\}$ associated with eigenvalues $0>\lambda_1\geqslant\lambda_2\geqslant\cdots$. Moreover, $\{u_k\}_{k=1}^{\infty}$ form a complete set in $L^2(\Lambda)$.

Lemma 2.13. A linear space generated by functions $[\exp(it\hat{L}^b)]\,f$, where $t\in\mathbb{R}$ and $f=\bar{f}$, $f\in L^2(\Lambda)$, is dense in $L^2(\Lambda)$.

Proof. It is enough to show that for every

$$f=\sum_{k=1}^{n} z_k u_k, \qquad z_k\in\mathbb{C}$$

there exist $t_0, t_1,\ldots, t_m\in\mathbb{R}$, $f_0=\bar{f}_0$, $f_1=\bar{f}_1,\ldots, f_m=\bar{f}_m$ from $L^2(\Omega)$ such that

$$f=\sum_{j=0}^{m}[\exp(it_j\hat{L}^b)]\,f_j$$

We exploit the fact that $\exp(it\hat{L}^b) = \sum_{k=1}^{\infty} [\exp(it\lambda_k)] P_k$, where P_k is the one-dimensional projector onto u_k.

Let $z_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$. Let us define $t_0 = 0$, $f_0 = \sum_{k=1}^{n} a_k u_k$, $m = 2n$, $t_j = -t_{j+n}$ for $j = 1,..., n$ and

$$f_j = \begin{cases} \frac{1}{2}\frac{b_j}{\lambda_j} u_j & \text{for } j = 1,..., n \\ -\frac{1}{2}\frac{b_{j-n}}{\lambda_{j-n}} u_{j-n} & \text{for } j = n+1,..., m \end{cases}$$

Then

$$\sum_{j=0}^{2n} [\exp(it_j\hat{L}^b)] f_j = \sum_{k=1}^{n} a_k u_k + \tfrac{1}{2} \sum_{j=1}^{n} \frac{b_j}{\lambda_j} [\exp(it_j\hat{L}^b) - \exp(-it_j\hat{L}^b)] u_j$$

but

$$\exp(it_j\hat{L}^b) - \exp(-it_j\hat{L}^b) = 2i \sum_{k=1}^{\infty} (\sin t_j\lambda_k) P_k$$

So by putting $t_j = (\pi/2)(1/\lambda_j)$ we obtain that

$$\sum_{j=0}^{2n} [\exp(it_j\hat{L}^b)] f_j = \sum_{k=1}^{n} a_k u_k + i \sum_{j=1}^{n} b_j u_j = \sum_{k=1}^{n} z_k u_k \quad \blacksquare$$

In the sequel we shall need also the following Feynman–Kac formulas:

Proposition 2.14. Let $\mathscr{E}(p) = p^2$, and let $0 < z < 1$.

1. For any $f = \bar{f} \in L_2(\Lambda)$, $b \in C_+(\partial\Lambda)$

$$\frac{\mathrm{Tr}_{\Gamma_{-1}(L^2(\Lambda))}\, e^{i\varphi_\Lambda(f)}\Gamma_{-1}(e^{-\beta(\Delta_\Lambda^b + \mu 1_\Lambda)})}{\mathrm{Tr}_{\Gamma_{-1}(L^2(\Lambda))}(\Gamma_{-1}(e^{-\beta(\Delta_\Lambda^b + \mu 1_\Lambda)}))}$$
$$\equiv \omega_0^{(\Lambda, b)}(W_F(f)) \equiv E e^{i\langle \xi_t^{(\Lambda, b)}, f\rangle}$$
$$= \mu_0^{(\Lambda, b)}(e^{i\langle \phi, \delta_0 \otimes f\rangle}) \tag{2.89}$$

2. For any $-\beta/2 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant \beta/2$, $f_1 = \bar{f}_1,..., f_n = \bar{f}_n \in L_2(\Lambda)$

$$\frac{\mathrm{Tr}_{\Gamma_{-1}(L^2(\Lambda))}(\alpha_{i\tau_1}^{(\Lambda, b)}({}^E\pi_0(W^F(f_1))) \cdots {}^E\alpha_{i\tau_n}^{(\Lambda, b)}({}^E\pi_0(W^F(f_n))))\, \Gamma_{-1}(e^{-\beta(\Delta_\Lambda^b + \mu 1_\Lambda)})}{\mathrm{Tr}_{\Gamma_{-1}(L^2(\Lambda))}(\Gamma_{-1}(e^{-\beta(\Delta_\Lambda^b + \mu 1)}))}$$

$$= E\left(\prod_{i=1}^{n} \exp(i\langle \xi_{\tau_i}^{(\Lambda, b)}, f_i\rangle)\right)$$

$$= \mu_0^{(\Lambda, b)}\left(\prod_{l=1}^{n} \exp(i\langle \phi, \delta_{\tau_i} \otimes f_i\rangle)\right) \tag{2.90}$$

3. For any sequence $(\Lambda_m, b_{\partial\Lambda_m})$ as in Lemma 2.12, any $-\beta/2 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant \beta/2$, $f_1,..., f_n \in L^2(\mathbb{R}^d)$ real and sufficiently small z, the limits

$$\lim_{m \to \infty} E\left(\prod_{i=1}^{n} \exp(i\langle \xi_{\tau_i}^{(\Lambda_m, b_m)}, f_i\rangle)\right)$$

$$\left[\text{resp. } \lim_{m \to \infty} \mu_0^{(\Lambda_m, b_{\partial\Lambda_m})}\left(\prod_{i=1}^{m} \exp(i\langle \phi, \delta_{\tau_i} \otimes f_i\rangle)\right)\right]$$

exist and are equal to

$$E\left(\prod_{l=1}^{m} \exp(i\langle \xi_{\tau_l}^{0}, f_l\rangle)\right)$$

$$\left[\text{resp. } \mu_0^{\beta}\left(\prod_{l=1}^{m} \exp(i\langle \phi, \delta_{\tau_l} \otimes f_l\rangle)\right)\right]$$

3. GENTLE PERTURBATIONS OF THE FREE BOSE GAS: THERMODYNAMIC LIMITS ON THE ABELIAN SECTOR

We shall study the thermodynamic limits of the multiplicative-like perturbations of the free thermal field $\mu_0^{(\beta,\mu)}$ given by the following perturbations:

$$\mu_{\Lambda,\varepsilon}^{(\beta,\mu)}(d\phi) = Z_\Lambda^{-1} \exp W_\Lambda(\phi_\varepsilon)\, \mu_0^{(\beta,\mu)}(d\phi) \tag{3.1}$$

where the interactions $W_\Lambda(\phi_\varepsilon)$ will be of the following form:

(LGP) The local gentle perturbations

$$W_\Lambda^L(\phi_\varepsilon) = \lambda \int d\rho(\alpha) \int_0^\beta d\tau \int_\Lambda :e^{i\alpha\phi_\varepsilon(\tau,x)}: dx \tag{3.2}$$

where

$$:e^{i\alpha\phi_\varepsilon(\tau,x)}: = \exp\frac{\alpha^2}{2} S_\varepsilon^\beta(0,x) \exp i\alpha\phi_\varepsilon(\tau,x)$$

$d\rho$ is a complex, bounded measure with a compact support and such that $\overline{d\rho(\alpha)} = d\rho(-\alpha)$; $\phi_\varepsilon(\tau,x) = (\phi * \chi_\varepsilon)(\tau,x)$, where $(\chi_\varepsilon)_\varepsilon > 0$ is a positive mollifier, i.e., $0 \leqslant \chi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, with support of size smaller than ε and such that $\int_\lambda \chi_\varepsilon(x)\, dx = 1$; λ is the strength of the perturbation.

(nLGP) Thee nonlocal gentle perturbations

$$W^{\rm nl}_\Lambda(\phi_\varepsilon) = \lambda \int_0^\beta d\tau \int d\rho(\alpha)\, d\rho(\alpha')$$
$$\times \int_\Lambda dx \int_\Lambda dy :e^{i\alpha\phi_\varepsilon(\tau, x)}: V(x-y): :e^{i\alpha\phi_\varepsilon(\tau, y)}: \tag{3.3}$$

where λ, $d\rho$, ϕ_ε are as in the local case, and the kernel V is chosen to be an L_1 integrable function.

Lemma 3.1. For both choices (LGP) and (nLGP) the thermodynamic stability bound

$$Z_\Lambda = \int d\mu_0^{(\beta,\mu)} \exp W_\Lambda(\phi_\varepsilon) \leqslant \exp C \cdot |\Lambda| \tag{3.4}$$

holds, where C is some constant depending on the details of the perturbations.

Proof. We shall consider only the case (nLGP). By simple Gaussian calculations we obtain

$$\int d\mu_0^{(\beta,\mu)}(\phi)\, W^{\rm nl}_\Lambda(\phi_\varepsilon)^n$$
$$= \lambda^n \int_0^\beta d\tau|_1^n \int d\rho(\alpha)_1^n \int d\rho(\alpha')_1^n \int_\Lambda dx|_1^n \int_\Lambda dy|_1^n$$
$$\times \prod_{i=1}^n V(x_i - y_i) \exp -\tfrac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^n \alpha_i\alpha_j S_\varepsilon^\beta(\tau_i-\tau_j, x_i-x_j)$$
$$\times \exp -\tfrac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^n \alpha'_i\alpha'_j S_\varepsilon^\beta(\tau_i-\tau_j, y_i-y_j)$$
$$\times \exp -\tfrac{1}{2} \sum_{i,j=1}^n \alpha_i\alpha'_j S_\varepsilon^\beta(\tau_i-\tau_j, x_i-y_j) \tag{3.5}$$

Using the positive-definitness of S_ε^β, we can estimate

$$\left|\int d\mu_0^{(\beta,\mu)}(\phi)\, W^{\rm nl}_\lambda(\phi_\varepsilon)^n\right|$$
$$= |\lambda|^n \beta^n (\operatorname{Var}\rho)^{2n} \exp(2nS_\varepsilon^\beta(0))\, \|V\|_1^n\, |\Lambda|^n \tag{3.6}$$

which shows the bound (3.4) with

$$C = |\lambda|\ \beta(\mathrm{Var}\ \rho)^2 \exp(2S^{\beta}_{\varepsilon}(0))\ \|V\|_1 \tag{3.7}$$

Moreover, it follows that Z_Λ are entire functions of the coupling constant $\lambda \in \mathbb{C}$. ∎

The characteristic functionals of the perturbed measures $\mu^{(\beta,\mu)}_{\Lambda,\varepsilon}$ can be written in the following forms:

(LGP) case:

$$\begin{aligned}
&\mu^{(\beta,\mu)}_{\Lambda,\varepsilon}(e^{i\langle\phi,\, g\otimes f\rangle}) \\
&\quad = \exp -\frac{1}{2} S^{\beta}_{0}(g\otimes f \mid g\otimes f) \sum_{n\geqslant 0} \frac{1}{n!} \int_0^{\beta} d\tau\,|^n_1 \int d\rho(\alpha)\,|^n_1 \\
&\qquad \times \int_\Lambda dx\,|^n_1 \prod_{i=1}^{n} [\exp\{-i\alpha_i (g\otimes f) * S^{\beta}_{\varepsilon}(\tau_i, x_i)\} - 1]\ \rho_{\Lambda,\varepsilon}(\tau, x)^n_1
\end{aligned} \tag{3.8}$$

where

$$\rho_{\Lambda,\varepsilon}(\tau,\alpha,x)|^n_1 = \lambda^n \int d\mu^{(\beta,\mu)}_{\Lambda}(\phi) \prod_{l=1}^{n} :e^{i\alpha_l \phi_\varepsilon(\tau_l, x_l)}: \tag{3.9}$$

(nLGP) case:

$$\begin{aligned}
&\mu^{(\beta,\mu)}_{\Lambda,\varepsilon}(e^{i\langle\phi,\, g\otimes f\rangle}) \\
&\quad = \exp -\frac{1}{2} S^{\beta}_{0}(g\otimes f \mid g\otimes f) \sum_{n\geqslant 0} \frac{1}{n!} \int_0^{\beta} d\tau\,|^n_1 \int d\rho(\alpha)\,|^n_1 \\
&\qquad \times \int dy\,|^n_1 \int d\rho(\alpha'_i)|^n_1 \int d(y_i)^n_1 \prod_{l=1}^{n} V(x_i - y_i) \\
&\qquad \times \prod_{i=1}^{n} [\exp\{-i\alpha_i(g\otimes f) * S^{\beta}_{\varepsilon}(\tau_i, x_i) - \beta_i (g\otimes f) * S^{\beta}_{\varepsilon}\} - 1] \\
&\qquad \times \sigma_{\Lambda,\varepsilon}(\tau, (\alpha, x)^n_1, (\beta, y)^n_1)
\end{aligned} \tag{3.10}$$

where

$$\sigma_{\Lambda,\varepsilon}(\tau^n_1, (\alpha, x)^n_1, (\beta, y)^n_1) = \lambda^n \mu^{(\beta,\mu)}_{\Lambda}\left(\prod_{l=1}^{n} :e^{i\alpha_l\phi_\varepsilon(\tau_l, x_l)}: \prod_{l=1}^{n} :e^{i\beta_l\phi_\varepsilon(\tau_l, y_l)}:\right) \tag{3.11}$$

Employing the integration by parts formula, we obtain the following equalities:

(LGP)

$$\rho_{\Lambda,\varepsilon}(\tau, x)_1^n = \lambda^n \exp - \sum_{i=2}^{n} S_\varepsilon^\beta(\tau_1 - \tau_i | x_1 - x_i)\, \alpha_i \alpha_1$$
$$\times \mu_{\Lambda,\varepsilon}^{(\beta,\mu)}\left(\prod_{l=2}^{n} :\exp[i\alpha_l \phi_\varepsilon(\tau_l, x_l)]: \exp\left\{\lambda \int_0^\beta d\tau \int_\Lambda dx \int d\rho(\alpha)\right.\right.$$
$$\times [\exp\{-\alpha\alpha_1 S_\varepsilon^\beta(\tau_1 - \tau; x_1 - x)\} - 1]$$
$$\left.\left. :\exp[i\alpha\phi_\varepsilon(\tau, x)]: \right\}\right) \tag{3.12}$$

(nLGP)

$$\sigma_{\Lambda,\varepsilon}((\tau, \alpha, x)_1^n, (\tau, \beta, y)_1^n)$$
$$= \lambda^n \exp - \sum_{i=2}^{n} \alpha_1 \alpha_i S_\varepsilon^\beta(\tau_1 - \tau_i | x_1 - x_i)$$
$$\times \exp - \sum_{i=2}^{n} \beta_1 \beta_i S_\varepsilon^\beta(\sigma_1 - \sigma_i | y_1 - y_i)$$
$$\times \mu_{\Lambda,\varepsilon}^{(\beta,\mu)}\left(\prod_{l=2}^{n} :\exp[i\alpha_l \phi_\varepsilon(\tau_l, x_l)]: \prod_{l=2}^{n} :\exp[i\beta_l \phi_\varepsilon(\sigma_l, y_l)]:\right.$$
$$\times \exp\left\{\lambda \int_0^\beta d\tau \int d\lambda(\alpha)\, d\lambda(\beta) \int_\Lambda dx \int_\Lambda dy\right.$$
$$\times [\exp\{-\alpha\alpha_1 S_\varepsilon^\beta(\tau_1 - \tau, x_1 - x)\} \exp\{-\beta\beta_1 S_\varepsilon^\beta(\tau_1 - \tau, y_1 - y)\} - 1]$$
$$\left.\left. \times :\exp[i\alpha\phi_\varepsilon(\tau, x)]: V(x - y) :\exp[i\beta\phi_\varepsilon(\tau, y)]: \right\}\right) \tag{3.13}$$

in which after a convergent expansion in powers of λ we recognize the well-known[22] Kirkwood–Salsburg-like equalities that hold between the correlation functions. A straightforward application of the contraction map principle[22] or the methods of the dual pairs of Banach spaces[38] leads to the proof of the following proposition in the (LGP) case.

Proposition 3.2 (LGP). 1. For $|\lambda| < \lambda_0(\text{LGP})$, where

$$\lambda_0(\text{LGP}) = \exp(-\alpha_*^2 S_\varepsilon(0, 0) - 1)\, C_\varepsilon^{l-1} \tag{3.14}$$

where

$$C_\varepsilon^l \equiv \sup_{\alpha'} \int_0^\beta d\tau \int d\,|\lambda|\,(\alpha) \int dx\, |e^{\alpha\alpha' S_\varepsilon(\tau, x)} - 1|$$
$$\alpha_*^2 = \sup\{\alpha^2 \in \operatorname{supp} d\lambda\}$$

the unique thermodynamic limits

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \rho_{\Lambda,\varepsilon}(\tau, \alpha, x)_1^n \equiv \rho_\varepsilon(\tau, \alpha, x)_1^n$$

exist in the sense of locally uniform convergence. The limiting correlation functions $\rho_\varepsilon(\tau, \alpha, x)_1^n$ are continuous, translationally invariant, and have the cluster decomposition property. Moreover, they are analytic functions in λ for $|\lambda| < \lambda_0(\text{LGP})$.

2. Let

$$\lambda \in \{z \mid z^{-1} \notin \sigma_\xi(\mathscr{K})\} \cap \{|z| < \xi\}$$

where K is the corresponding infinite-volume KS-operator, and $\sigma_\xi(K)$ is the spectrum of K in the corresponding Banach space B_ξ (compare refs. 38 and 39).

Then for any such λ the unique thermodynamic limits

$$\rho_\varepsilon(\tau, \alpha, x)_1^n = \lim_{\Lambda \uparrow \mathbb{R}^d} \rho_{\Lambda,\varepsilon}(\tau, \alpha, x)_1^n$$

exist in the sense of locally uniform convergence and are analytic functions in λ.

As a simple corollary we obtain:

Proposition 3.3 (LGP). 1. For $\lambda \in \mathbb{C}$ as described in point 1 or 2 of Proposition 3.2 the weak limit $d\mu_\varepsilon^\lambda$ of the measure $d\mu_{\Lambda,\varepsilon}^\beta$ exists and the limiting measure $d\mu_\varepsilon^\lambda$ is periodic in β, symmetric on K_β, and OS-positive on K_β. Moreover, $d\mu_\varepsilon^\lambda$ is (weakly) analytic in the λ perturbation of the free measure $d\mu_0^\beta$.

2. For $|\lambda| < \lambda_0(\text{LGP})$ the limiting measure $d\mu_\varepsilon^\lambda$ is translationally invariant with respect to the translations of $\mathbb{R}^d$ and is ergodic under the action of this group.

3. For λ as in part 1 the characteristic functional of $d\mu_\varepsilon^\lambda$ is given by

$$\begin{aligned}
\mu_\varepsilon^\lambda(e^{i(\phi, g\otimes f)}) = \exp -\frac{1}{2} S_0^\beta(g\otimes f \mid g\otimes f) \\
\times \sum_{n\geqslant 0} \frac{1}{n!} \int d\rho(\alpha)\, d\tau\, dx \,|_1^n \\
\times \prod_{l=1}^{n} [\exp\{-\alpha_l S_\varepsilon^\beta * (g\otimes f)(\tau_l, x_l)\} - 1] \\
\times \rho_\varepsilon(\tau, \alpha, x)_1^n
\end{aligned} \tag{3.15}$$

A minor modification of the original analysis of the Kirkwood–Salsburg identities enables us to control also the thermodynamic limits for a nonlocal gentle perturbation (3.3).

Proposition 3.4 (nLGP). Let $W = (\mathrm{nGLP})$.

1. For $\lambda \in \mathbb{C}$: $|\lambda| < \lambda_0(\mathrm{nLGP})$, where

$$\lambda_0(\mathrm{nLGP}) \equiv \exp(-2\alpha_*^2 S_\varepsilon(0,0) - 1)(C_\varepsilon^{\mathrm{nl}})^{-1}$$

$$C_\varepsilon^{\mathrm{nl}} \equiv \sup_{\gamma, \gamma'} \int_0^\beta d\tau \int d|\lambda|(\alpha) \int d|\lambda|(\alpha') \times \int dx \int dy\, V(x-y)\, |e^{-\alpha\gamma S_\varepsilon(\tau,x)} e^{-\alpha'\gamma' S_\varepsilon(\tau,y)} - 1|$$

$$\alpha_* = \sup\{\alpha \in \operatorname{supp} d\lambda\}$$

$\lim d\mu_{\Lambda,\varepsilon}^{\beta,\mu} = d\mu_\varepsilon^\lambda$ exists and the limiting measure $d\mu_\varepsilon^\lambda$ is periodic in β, symmetric on K_β, and OS-positive on K_β. Moreover, $d\mu_\varepsilon^\lambda$ is (weakly) analytic in the λ perturbation of the free measure. The measure $d\mu_\varepsilon^\lambda$ is $E(d)$ invariant and ergodic under the translations by $\mathbb{R}^d$. The characteristic functional of $d\mu_\varepsilon^\lambda$ is given by

$$\begin{aligned}\mu_\varepsilon^\lambda(e^{i(\phi, g\otimes f)}) = {} & \exp -\frac{1}{2} C_0^\beta(g\otimes f \mid g\otimes f) \\ & \times \sum_{n\geq 0} \frac{1}{n!} \int d(\tau, x, \alpha)_1^n\, d(\tau', x', \alpha')_1^n \prod_{i=1}^n V(x_i' - x_i) \\ & \times \prod_{i=1}^n [\exp(-\alpha_i S_\varepsilon^\beta * (g\otimes f)(\tau_i, x_i)) \\ & \times \exp(-\alpha_i' S_\varepsilon^\beta * (g\otimes f)(\tau_i', x_i')) - 1] \\ & \times \sigma_\varepsilon^\lambda((\tau, x, \alpha)_1^n; (\tau', x', \alpha')_1^n)\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\sigma_\varepsilon^\lambda((\tau, x, \alpha)_1^n; (\tau', x', \alpha')_1^n) & \equiv \lim_{\Lambda \uparrow \mathbb{R}^d} \mu_{\Lambda,\varepsilon}^\lambda \left(\prod :e^{i\alpha_i \phi_\varepsilon(\tau_i, x_i)}: \prod_{i=1}^n :e^{i\alpha_i' \phi_\varepsilon(\tau_i', x_i')}: \right) \\ & = \mu_\varepsilon^\lambda \left(\prod_{i=1}^n :e^{i\alpha_i \phi_\varepsilon(\tau_i, x_i)}: \prod_{i=1}^n :e^{i\alpha_i' \phi_\varepsilon(\tau_i', x_i')}: \right)\end{aligned} \tag{3.17}$$

In particular, we have obtained the following functional integral representation of the corresponding multitime Euclidean Green functions corresponding to the infinite-volume limit perturbations of the free Bose gas in the noncritical regime.

Theorem 3.5. Let $V_\Lambda = \text{(LGP)}$ or $V_\Lambda = \text{(nLGP)}$ and $\lambda \in \mathbb{C}$ be restricted as in part 1 of Proposition 3.4 or part 2 of Proposition 3.2 in the (LGP) case. Then the Euclidean multitime Green functions on $\mathscr{A}(\mathbf{h})$ are given by the following functional integrals:

$$
\begin{aligned}
&{}^E G^\lambda_{f_1,\dots,f_n}(s_1,\dots,s_n)\\
&\quad \equiv \lim_{\Lambda\uparrow\infty} {}^E G^{\lambda,\Lambda}_{f_1,\dots,f_n}(s_1,\dots,s_n)\\
&\quad = \int_{\mathscr{D}'(K_\beta\times\mathbb{R}^d)} d\mu^\lambda_\varepsilon(\phi) \prod_{i=1}^n \exp(i\langle\phi;\delta_{s_i}\otimes f_i\rangle)\\
&\quad \overset{\text{(LGP)}}{=} {}^E G^0_{f_1,\dots,f_n}(s_1,\dots,s_n)\\
&\qquad \times \sum_{n\geqslant 0}^{\infty} \frac{1}{n!} \int d(\tau,x,\alpha)_1^n \prod_{i=1}^n \left\{ \exp -\alpha_i S^\beta_\varepsilon * \left(\sum_{l=1}^n \delta_{\bar{s}_l}\otimes f_i\right)(x_i) - 1 \right\}\\
&\qquad \times \rho_\varepsilon((\tau,x,\alpha)_1^n)\\
&\quad \overset{\text{(nLGP)}}{=} {}^E G^0_{f_1,\dots,f_n}(s_1,\dots,\sigma_n)\\
&\qquad \times \sum_{n\geqslant 0}^{\infty} \int d(\tau,x,\alpha)_1^n\, d(\tau',x',\alpha)_1^n \prod_{i=1}^n V(x_i - x'_i)\\
&\qquad \times \left[\exp - S^\beta_\varepsilon * \left(\sum_{l=1}^n \delta_{\bar{s}_l}\otimes f_i\right)(x) - \alpha' S^\beta_\varepsilon * \left(\sum_{l=1}^n \delta_{\bar{s}_l}\otimes f_i\right)(x') - 1 \right]\\
&\qquad \times \mu^\lambda_{\Lambda,\varepsilon}\left(\prod_{l=1}^n {:}\exp i\alpha_l \phi_\varepsilon(\tau_l, x_l){:} \cdot {:}\exp i\alpha'_l \phi_\varepsilon(\tau'_l, x'_l){:} \right)
\end{aligned}
\tag{3.18}
$$

Some properties of the system that are elementary albeit fundamental for the purposes of the present paper are collected in the following proposition:

Proposition 3.6. Let $\{{}^E G^\lambda_{f_1,\dots,f_n}(s_1,\dots,\sigma_n)\}$ be a collection of the Euclidean multitime infinite-volume Green functions constructed in Theorem 3.5. Then they can be extended by continuity to the Abelian sector $\mathscr{A}(\mathbf{h})$ of the Weyl algebra $\mathscr{W}(\mathbf{h})$, and the continued Green functions denoted by the same symbol obey properties EG(1)–EG(5)(i).

Corollary 3.7. Let $|\lambda| < \lambda_0(\text{LGP})$ [for the case (LPG)] and $|\lambda| < \lambda_0(\text{nLGP})$ [for the case (nLGP)]. Then the following perturbation expansions are convergent:

$$
\begin{aligned}
&{}^{E}G^{\lambda}_{f_1,\dots,f_n}(s_1,\dots,s_n)\} \\
&\overset{(\text{LGP})}{=} \sum_{n\geqslant 0} \frac{\lambda^n}{n!} \int \cdots \int_{K_\beta \times \mathbb{R}^d \times \mathbb{R}} d\tau\, dx\, d\rho(\alpha)|_1^n \\
&\quad \times \left\langle \prod_{l=1}^{n} \exp\langle \phi, \delta_{s_l} \otimes f_l \rangle; :\exp[i\alpha_1 \phi_\varepsilon(\tau_1, x_1)]: ; \cdot ; \cdot \right\rangle_0^{\beta, T} \qquad (3.19)
\end{aligned}
$$

where $\langle \cdot ; \cdot ; \cdot ; \cdot \rangle_0^{\beta, T}$ denote the truncated expectation values with respect to the free gas measure $d\mu_0^\beta$.

For a class of gentle perturbations of the free Bose gas stochastic structure another variety of existence results can be established using the methods of refs. 5 and 40. For this let us assume now that our perturbations are of the following forms:

$(\text{LGP})_e \quad W_\Lambda(\phi_\varepsilon) = (3.2)$

but now $d\rho$ is an even bounded real measure, $\lambda \geqslant 0$, or

$(\text{nLGP})_e \quad W_\Lambda(\phi_\varepsilon) = (3.3)$

where $d\rho$ is also an even bounded real measure and $V \in L_1(\mathbb{R}^d)$ is assumed to be pointwise nonnegative, i.e., $V(x) \geqslant 0$ and $\lambda \geqslant 0$.

Proposition 3.8. Let $d\mu^\lambda_{\Lambda,\varepsilon}$ be a locally perturbed free Bose gas measure by $(\text{LGP})_e$ or $(\text{nLGP})_e$ and let $\lambda > 0$. Then the following correlation inequalities of the Fröhlich–Park type are valid:

1. $Z_{\Lambda_1 \cup \Lambda_2} \geqslant Z_{\Lambda_1} \cdot Z_{\Lambda_2} \qquad (3.20)$
2. $\langle \phi^2(g \otimes f); :\cos \alpha\phi_\varepsilon: (\tau, x) \rangle^{\lambda, T}_{\Lambda, \varepsilon} \leqslant 0 \qquad (3.21)$
3. $\left\langle e^{i\phi(g\otimes f)}; \prod_{i=1}^{n} :\cos \alpha_i \phi_\varepsilon: (\tau_i, x_i) \right\rangle^{\lambda, T}_{\Lambda, \varepsilon} \leqslant 0 \qquad (3.22)$
4. $\left\langle e^{i\phi(g\otimes f)}; \prod_{i=1}^{n} :\cos \alpha_i \phi_\varepsilon: (\tau_i, x_i) \right\rangle^{\lambda, T}_{\Lambda, \varepsilon} \geqslant 0 \qquad (3.23)$
5. $\left\langle \prod_i \cos \alpha_i \phi_\varepsilon(s_i, x_i) \prod_j \cos \beta_j \phi_\varepsilon(t_j, y_j) \right\rangle^{\lambda, T}_{\Lambda, \varepsilon} \geqslant 0 \qquad (3.24)$

Proof. Basically the same as in ref. 5, employing the duplicate variable trick and elementary trigonometric identities. ∎

Theorem 3.9. Let us consider perturbation $(\mathrm{LGP})_e$ or $(\mathrm{nLGP})_e$ of the free Bose gas thermal field $d\mu_0^\beta$.

1. For any $\lambda \geqslant 0$ the unique thermodynamic limit

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \mu_{\Lambda,\varepsilon}^{\lambda}\left(\prod_{i=1}^{n} \exp(i\langle \phi, \delta_{s_i} \otimes f_i \rangle)\right)$$
$$\equiv \mu_\varepsilon^\lambda \left(\prod_{i=1}^{n} \exp(i\langle \phi, \delta_{s_i} \otimes f_i \rangle)\right)$$
$$\equiv {}^E G^\lambda_{f_1,\dots,f_n}(s_1,\dots,s_n) \qquad \text{for} \quad -\beta/2 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant \beta/2 \quad (3.25)$$

exists and the limiting Green functions obey all the properties EG(1)–EG(5)(i).

2. In particular the following estimates hold:

(a) $$\left| S_\lambda^2(f\otimes g \mid f\otimes g) \equiv \frac{d^2}{i^2\, d\alpha_1\, d\alpha_2}\, {}^E G^\lambda_{\alpha_1 g, \alpha_2 g}(f, f)\Big|_{\substack{\alpha_1 = 0\\ \alpha_2 = 0}} \right|$$
$$\leqslant S_0^\beta(f\otimes g \mid f\otimes g) \qquad (3.26)$$

(b) $$\left| \mu_\varepsilon^\lambda \left(\exp S \int_0^\beta d\tau\, f(\tau) \int dx\, g(x)\, \phi(\tau, x) \right) \right|$$
$$\leqslant \exp \operatorname{Re} \frac{S^2}{2} S_0^\beta(f\otimes g \mid f\otimes g) \qquad (3.27)$$

(c) $$\left| S_\lambda^{n,\beta}(f_1\otimes g_2,\dots, f_n\otimes g_n) \equiv \mu_\varepsilon^\lambda \left(\prod_{i=1}^n \langle \phi, f_i \otimes g_i \rangle \right) \right|$$
$$\leqslant \sigma(n!)^{1/2} \prod_{i=1}^n |S_0^\beta(f_i\otimes g_i \mid f_i \otimes g_i)| \qquad (3.28)$$

Proof. From the correlation inequality (3.23) it follows that $\mu_\Lambda^\lambda(e^{i\phi(f\otimes g)})$ monotonously increases in the volume and that for real t, $\mu_\Lambda^\lambda(e^{t\phi(f\otimes g)})$ decreases as $\Lambda \uparrow \mathbb{R}^d$. This leads to the statement that the unique limit $\lim_\Lambda \mu_\Lambda(e^{\zeta(\phi, f\otimes g)}) \equiv \mu_\infty^\lambda(e^{\zeta(\phi, f\otimes g)})$ exists and obeys the estimate (3.27). Then the estimates (3.28) follow by the application of the Cauchy integral formula and the analycity in ζ of $\mu_\infty^\lambda(e^{\zeta(\phi, f\otimes g)})$. Although the estimate (3.26) follows from (3.28), its independent proof follows easily from the correlation inequality (3.21), which says that $\mu_\Lambda^\lambda(\phi, f\otimes g)^2$ is monotonously decreasing in the volume.

Integrating by parts on the functional space $\mathscr{D}'(K_\beta \times \mathbb{R}^d)$ with respect to the measure $d\mu_\Lambda^\lambda(\phi)$, we obtain

$$
\begin{aligned}
{}^{E}G^{\lambda}_{\Lambda, f_1,\dots,f_n}(s_1,\dots,s_n) \underset{\text{GLP}_\varepsilon}{=} {}^{E}G^{0}_{f_1,\dots,f_n}(s_1,\dots,s_n) \\
\times \sum_{k\geqslant 0}^{\infty} \frac{\lambda^k}{k!} \int_{\Lambda\times K_\beta\times\mathbb{R}} d\tau\, dx\, d\lambda(\alpha)|_1^k \\
\times \prod_{i=1}^{k} \left[\exp\left\{ -\sum_{j=1}^{n} \alpha_j S_\varepsilon^\beta * (\delta_{s_j}\otimes f_j)\right\} - 1\right] \\
\times \mu_\Lambda^\lambda \left(\prod_{j=1}^{n} :\exp[i\alpha_j\phi_\varepsilon(\tau_j, x_j)]: \right)
\end{aligned} \tag{3.29}
$$

From the correlation inequality (3.24) it follows that

$$
\mu_\Lambda^\lambda \left(:\prod_{i=1}^{n} \cos\alpha_i\phi_\varepsilon(\tau_i, x_i): \right) \equiv C_\Lambda^\lambda(\alpha_1, \tau_i, x_i|_1^n) \tag{3.30}
$$

monotonously increase in the volume Λ and because they are uniformly bounded

$$
|C_\Lambda^\lambda(\tau_i, x_i|_1^n)| \leqslant \exp \tfrac{1}{2}\beta^2 n C_\varepsilon^\beta(0) \tag{3.31}
$$

the unique thermodynamic limit $\lim_\Lambda C_\Lambda^\lambda \equiv C^\lambda$ exists pointwise on $(K_\beta \times \mathbb{R}^d)^{\otimes n}$. From this, the existence of pointwise limits

$$
\lim_\Lambda \mu_\Lambda^\lambda \left(\prod_{j=1}^{n} :e^{i\alpha_j\phi_\varepsilon(\tau_1, x_j)}: \right) = \mu_\varepsilon^\lambda \left(\prod_{j=1}^{n} :e^{i\alpha_j\phi_\varepsilon(\tau_j, x_j)}: \right) \tag{3.32}
$$

follows in the same way as demonstrated in ref. 40 by the application of another correlation inequality (originally due to Pfister[41]) not listed in Proposition 3.8 but formulated in ref. 40 in a similar context. Finally, the proven pointwise convergence is sharpened to the local uniform one by a standard application of the Mayer–Montroll identities (see, e.g., ref. 38). From the obtained convergence the following expression for the infinite-volume Euclidean Green functions ${}^{E}G^{\lambda}_{f_1,\dots,f_n}(s_1,\dots,s_n)$ follows easily from (3.29):

$$
\begin{aligned}
{}^{E}G^{\lambda}_{f_1,\dots,f_n}(s_1,\dots,s_n) = {}^{E}G^{0}_{f_1,\dots,f_n}(s_1,\dots,s_n) \\
\times \sum_{k\geqslant 0} \frac{1}{k!} \int_{\mathbb{R}_\beta\times\mathbb{R}^d\times\mathbb{R}} d\tau\, dx\, d\lambda(\alpha)|_1^k \\
\times \prod_{i=1}^{k} \left[\exp\left\{ -\sum_{j=1}^{n} \alpha_i S_\varepsilon^\beta * (\delta_{s_j}\otimes f_j)\right\} - 1\right] \\
\times \mu_\infty^\lambda \left(\prod_{i=1}^{n} :\exp[i\alpha_i\phi_\varepsilon(\tau_i, x_i)]: \right)
\end{aligned} \tag{3.33}
$$

The case of nLGP_ε is analyzed in a similar way. ∎

Remarks. The existence and uniqueness of the thermodynamic limits for the Euclidean Green functions ${}^{E}G^{\lambda}_{f_1,\dots,f_n}(s_1,\dots,s_n)$ follow easily from the correlation inequality (3.23) and the uniform bound

$$|{}^{E}G^{\lambda}_{f_1,\dots,f_n}(s_1,\dots,s_n)| \leqslant 1 \tag{3.34}$$

Using methods based on the analysis of the corresponding Kirkwood–Salsburg identities, one can study the gentle perturbations of the local, free, conditioned thermal fields described in Section 2.2.

For this goal let us consider a perturbation of the free, conditioned [by $b_{\partial\Lambda} \in C(\partial\Lambda)$] thermal field $\mu_0^{(\Lambda, b_{\partial\Lambda})}$ of the form

$$\tilde{\mu}^{(\lambda, b_{\partial\Lambda})}_{\Lambda,\varepsilon}(d\Phi) = \tilde{Z}^{-1}_{\Lambda,\varepsilon}(b_{\partial\Lambda}) \exp W_{\Lambda}(\Phi_{\varepsilon})\, d\mu_0^{(\Lambda, b_{\partial\Lambda})}(\Phi) \tag{3.35}$$

where

$$\tilde{Z}_{\Lambda,\varepsilon}(b_{\partial\Lambda}) = \mu_0^{(\Lambda, b_{\partial\Lambda})}(\exp W_{\Lambda}) \tag{3.36}$$

and $W_{\Lambda}(\Phi_{\varepsilon})$ is given by (3.2) or (3.3).

Theorem 3.10. Let (Λ_α) be any arbitrary net of bounded subsets of $\mathbb{R}^d$ with the boundaries of class at least C^3-piecewise. Additionally we shall require that the mean curvature of $\partial\Lambda_\alpha$ is uniformly bounded in α. Let $(b^{\alpha}_{\partial\Lambda_\alpha})$ be a sequence of continuous boundary conditions.

Then for $|\lambda| < \lambda_0(\mathrm{LGP})$, if $W_{\Lambda} = \mathrm{LGP}$ [respectively $|\lambda| < \lambda_0(\mathrm{nLGP})$, if $W_{\Lambda} = \mathrm{nLGP}$] the unique thermodynamic limit

$$\lim_{\alpha} \tilde{\mu}^{(\lambda, b_{\partial\Lambda_\alpha}}_{\Lambda,\varepsilon} \equiv \tilde{\mu}^{\lambda}_{\varepsilon}$$

exists in the sense of weak convergence and moreover $\tilde{\mu}^{\lambda}_{\varepsilon} = \mu^{\lambda}_{\varepsilon}$.

Proof. The method of the dual pair of Banach spaces as explained in ref. 38 and applied in a similar situation in refs. 39 and 40 is applied. ∎

Remark. The method of refs. 38–40 gives the existence and independence on the classical boundary conditions of the limiting thermal field $\tilde{\mu}^{\lambda}_{\varepsilon}$ in a larger set of λ (see also point 2 in Proposition 3.3).

As a corollary we have the following result:

Corollary 3.11. Let $(\Lambda_\alpha)_\alpha$, $(b_{\partial\Lambda_\alpha})_\alpha$ be as in Theorem 3.10 and let ${}^{\Lambda}\mathbb{G}_{\lambda}(\Lambda_\alpha, b_{\partial\Lambda_\alpha})$ be the system of the Euclidean Green functions corresponding to the gentle perturbations of the local, conditioned, free W^*-KMS structure restricted to the Abelian sector $\mathscr{A}(\mathbf{h}_{\Lambda})$ of $\mathscr{W}(\mathbf{h}_{\Lambda})$. Then for λ as in Theorem 3.10 and $0 < z < 1$ sufficiently small the unique thermodynamic limits of the corresponding Euclidean Green functions exist and are equal to those obtained in Theorems 3.5 and 3.9.

All the systems of limiting Euclidean Green functions constructed in this section obey properties EG(1)–EG(5)(i) and correspond to some generalized thermal processes.

Therefore the general reconstruction procedure of ref. 20 applies (see Proposition 2.9), leading to certain W^*-KMS structures. Further analysis of the derived W^*-KMS structures is contained in forthcoming papers.

4. CONCLUDING REMARKS

4.1. For the finite-volume perturbations of the free thermal field $\mu_0^{(\beta,\mu)}$ the corresponding nonhomogeneous process $(\xi_t^{(\lambda,\Lambda)})_{t\in K_\beta}$ has the two-sided Markov property on K_β in the sense of Proposition 2.11. The interesting and important question is whether the homogeneous limit $\Lambda\uparrow\mathbb{R}^d$ preserves the above Markov property. For a gentle perturbation of a class of lattice anharmonic crystals some results on the preservation of the two-sided Markov property in the thermodynamic limit have been established in ref. 42. A constructive route for the verification of the two-sided Markov property will be formulated below.

4.2. The notion of DLR equations for the gentle perturbations of the Abelian sector of the free Bose gas in the Euclidean region can be introduced. For this goal, let us denote by $\Pi(\Lambda^C)$ the orthogonal projector [in the space $\mathscr{H}_0^\beta\equiv \text{m.c.}(C(K_\beta)\times D(\mathbb{R}^d);S_0^\beta)$] onto the subspace $\mathscr{H}_0^\beta(\Lambda^C)\equiv \text{m.c.}(C(K_\beta)\times C_C^\infty(\Lambda^C);S_0^\beta)$, for $\Lambda\subset\mathbb{R}^d$ open and bounded.

The free thermal kernel S_0^β is then decomposed as

$$S_0^\beta={}^{\Lambda^C}S_0^\beta+{}^{\Lambda^C}\Pi_0^\beta \tag{4.1}$$

where

$${}^{\Lambda^C}S_0^\beta\equiv S_0^\beta\circ(1-\Pi(\Lambda^C));\qquad {}^{\Lambda^C}\Pi_0^\beta=S_0^\beta\circ\Pi(\Lambda^C) \tag{4.2}$$

Let $\mu_0^{\Lambda^C}$ be a Gaussian random field with the covariance given by ${}^{\Lambda^C}S_0^\beta$. It is clear that the symmetry and OS positivity on K_β of the free conditioned Gaussian random field $\mu_0^{\Lambda^C}$ is preserved and moreover $\mu_0^{\Lambda^C}\to\mu_0^\beta$ weakly as $\Lambda\uparrow\mathbb{R}^d$.

Let $\Sigma^0(\Lambda^C)$ be the (μ^0-complete) σ-algebra generated by the random elements of the form $\langle\Phi,f\rangle$, where $f\in\mathscr{H}_0^\beta(\Lambda^C)$. Then the conditional expectation values of the measure μ_0^β with respect to the σ-algebra $\Sigma^0(\Lambda^C)$ are given by

$$E_{\mu_0}\{F\,|\,\Sigma^0(\Lambda^C)\}(\Psi)=\mu_0^{\Lambda^C}(F(\cdot+\Pi^*_{\Lambda^C}(\Psi)) \tag{4.3}$$

for μ_0-a.e. $\Psi \in \mathscr{D}'(K_\beta \times \mathbb{R}^d)$, where

$$\langle \Pi^*_{\Lambda^C}(\Psi), f \rangle \equiv \langle \Psi, \Pi_{\Lambda^C}(f) \rangle \tag{4.4}$$

The corresponding conditional expectation values of the perturbed measure are

$$E_{\mu_{\Lambda,\varepsilon}}\{F \mid \Sigma^0_{\Lambda^C})\}(\Psi) = \frac{\mu_0^{\Lambda^C}(F(\cdot + \Pi^*_{\Lambda^C}(\Psi)) \exp W_\Lambda(\cdot + \Pi^*_{\Lambda^C}(\Psi))}{\mu_0^{\Lambda^C}(\exp W_\Lambda(\cdot + \Pi^*_{\Lambda^C}(\Psi))} \tag{4.5}$$

for μ_0-a.e. $\Psi \in \mathscr{D}'(K_\beta \times \mathbb{R}^d)$.

In analogy to ref. 27 (see also refs. 43 and 44) we define a classical thermal Gibbs measure corresponding to the gentle perturbation of the free Bose gas as any probabilistic, cylindric Borel measure μ on $\mathscr{D}'(K_\beta \times \mathbb{R}^d)$ such that

$$\mu \circ E_{\mu_{\Lambda,\varepsilon}}\{\Sigma(\Lambda^C)\} = \mu \tag{DLR}$$

for any open, bounded $\Lambda \subset \mathbb{R}^d$.

It is evident that any solution of (DLR) defines a thermal random field in the sense of Definition 2.8. Some results about the uniqueness of the solutions of (DLR) generalizing slightly Theorem 3.10 shall be reported elsewhere (see also refs. 39 and 40).

The introduced concept of the classical thermal Gibbs measure will be of particular interest in the case of polynomial perturbations where several solutions of the corresponding (DLR) equations may exist.[18]

Using the (DLR) equation, the constructive approach to the problem of preservation of the two-sided Markov property on the circle K_β for the limiting thermal random field μ^λ_ε can be formulated. The idea is to show that for μ^λ_ε-a.e. $\Psi \in \mathscr{D}'(\mathbb{R}^d)$ the limit

$$\lim_{\Lambda \uparrow \mathbb{R}^d} E_{\mu^\lambda_{\Lambda,\varepsilon}}\{F \mid \Sigma^0([t,s]^C \times \Lambda^C)\}(\Psi)$$

(where $\Sigma([t,s]^C \times \Lambda^C)$ is the σ-algebra generated by the random elements $\langle \Phi, g \otimes f \rangle$, with g supported on the segment $[t,s]^C$ and f supported in Λ^C) exists and is equal (μ^λ_ε-a.e.) to the conditional expectation value

$$E_{\mu^\lambda_\varepsilon}\{F \mid \Sigma(\{t,s\}^C)\}(\Psi)$$

Details of the proof that indeed, for small values of $|\lambda|$, this is true will be reported elsewhere.[18]

4.3. For a bounded $\Lambda \subset \mathbb{R}^d$ the theory of bounded perturbations of the KMS structures (see, e.g., ref. 17, Chapter 4, and references therein) can be applied in the thermal representation enabling us to study the gentle perturbations on the whole Weyl algebra. It is proven in ref. 18 that again the nonhomogeneous thermal process $(\xi_t^{\lambda, \Lambda})_{t \in K_\beta}$ determines the corresponding W^*-KMS structure obtained from the corresponding GNS representation. The important problems of constructing the perturbed (Euclidean-time) Green functions on the whole Weyl algebra $\mathscr{W}(\mathbf{h})$ and of whether the corresponding homogeneous process $(\xi_t^{\lambda})_{t \in K_\beta}$ determines them and also whether the limiting W^*-KMS structure on $\mathscr{W}(\mathbf{h})$ forms a modular structure will be treated in another paper in this series.

4.4. The Abelian sector of the free Bose critical gas can be described in the Euclidean region by a certain nonergodic Gaussian generalized thermal process. Results complementary to those contained in Section 2 for the critical gas are obtained in ref. 18, where thermodynamic limits of the gentle perturbations on the Abelian sector also have been controlled by applications of the Fröhlich–Park correlation inequalities. The most interesting result of these investigations is that nonergodicity of the limiting, perturbed thermal process is preserved. Whether this is connected to the preservation of the Bose–Einstein condensate in the interacting system remains to be answered.

4.5. More general, unbounded perturbations (e.g., of polynomial type) will be described in an another paper of this series.[18] Standard tools of constructive Euclidean quantum field theory, such as the high- (and the low-) temperature cluster expansions, are used to study the corresponding perturbations of the free thermal structure on the Abelian sector.

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